MODULES OVER UNRAMIFIED REGULAR LOCAL RINGS[1]

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Introduction

Throughout this paper we assume that all rings are commutative, noetherian rings with unit and that all modules are finitely generated and unitary. The main object of study is what it means about two modules A and B over an unramified regular local ring to assert that the torsion submodule of $A \otimes B$ is zero. The basic fact established (see §3) is that if $A \otimes B$ is torsion-free and not zero, then

(a) A and B are torsion-free,

(b) $\mathrm{Tor}_i^R(A, B) = 0$ for all $i > 0$, and

(c) $\mathrm{hd}\, A + \mathrm{hd}\, B = \mathrm{hd}\,(A \otimes B) < \dim R$,

where hd A means the homological dimension of A (we refer the reader to [1] for notation and basic homological facts used). Using this result we give the following criteria for a module A over an unramified regular local ring R of dimension n to be free: (a) The tensor product of A with itself n-times is torsion-free; (b) $A \otimes A \otimes \mathrm{Hom}(A, R)$ is torsion-free. Section 3 concludes with some results which seem to indicate that the module theory of odd-dimensional unramified regular local rings is different from the module theory of even-dimensional unramified regular local rings.

The proofs of most of the results, including those just mentioned, are based in an essential way on the fact established in §2 that for an unramified regular local ring R and a torsion-free R-module A if $\mathrm{Tor}_i^R(A, B) = 0$ for some R-module B, then $\mathrm{Tor}_j^R(A, B) = 0$ for all $j \geqq i$. In fact, if this property of Tor can be established for arbitrary regular local rings, then almost all the results of this paper extend immediately to all regular local rings.

1. Some properties of Tor

Before proceeding to the main results of this section we review briefly some of the basic facts concerning the codimension of a module as can be found for instance in [1] or [2].

Let R be a local ring with maximal ideal $\mathfrak{m}$ and A a nonzero R-module. A sequence of elements $x_1, \cdots, x_t$ in $\mathfrak{m}$ is called an A-sequence if x_1 is not a zero-divisor in A and x_i is not a zero-divisor for $A/(x_1, \cdots, x_{i-1})A$ for all $i = 2, \cdots, t$. If $x_1, \cdots, x_t$ is an A-sequence, then it is easily seen that $t \leqq \dim R$ (where dim R means the Krull dimension of R). Thus it makes sense to talk about maximal A-sequences. It can be shown that all maximal

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