BOREL STRUCTURES FOR FUNCTION SPACES[1]

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If X and Y are topological spaces, then Y^X denotes the set of all continuous mappings from X into Y. For a given topology on Y^X, we may ask whether the natural mapping $\varphi: Y^X \times X \to Y$ defined by $\varphi(f, x) = f(x)$ is continuous; if it is, then the topology on Y^X is said to be *admissible* [1]. It is always possible to find an admissible topology; for instance, the discrete topology on Y^X is always admissible. Moreover, when X is locally compact, Y^X has a unique smallest[2] admissible topology; this is the familiar "compact-open" topology. These and related questions concerning topologies for function spaces have been investigated in considerable detail by several authors [1, 4].

We are interested in the analogous situation when X and Y are Borel spaces[3] rather than topological spaces; in this case we define Y^X as the set of all Borel mappings[4] from X into Y. Unfortunately, it turns out that even for some of the simplest Borel spaces, it is impossible to define a Borel structure on Y^X so that φ is a Borel mapping; even if we impose the discrete structure on Y^X, φ will in general not be Borel. As a substitute, we may ask ourselves the following questions: For which *subsets* F of Y^X is it possible to impose a Borel structure on F so that $\varphi \mid F \times X$ will be Borel? If is is possible for a given F, what can we say about the appropriate structures? In particular, is there always a smallest such structure (corresponding to the compact-open topology)?

Let us introduce some terminology. We will write "space" instead of "Borel space", "structure" instead of "Borel structure", and φ_F instead of $\varphi \mid F \times X$. A structure R on F for which φ_F is Borel will be called *admissible*; a subset F of Y^X on which it is possible to impose an admissible structure is

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[1] The results proved here were announced in [2]. The author is grateful to Prof. P. R. Halmos for many helpful suggestions, and in particular for the counterexample in Section 7; also to Dr. M. Rabin for a number of helpful discussions.

[2] I.e., weakest, with fewest open sets. We remark that the local compactness condition on X may be replaced by certain other conditions on X and Y; see [1, 4].

[3] A *Borel space* is a set X together with a σ-ring of subsets of X called *Borel sets* whose union is all of X. The σ-ring of Borel sets is called the *Borel structure* of X, or simply its *structure*. The structure of the cartesian product of two Borel spaces X and Y is taken to be that generated by the *Borel rectangles*—the products of a Borel set in X and a Borel set in Y. Our definition of Borel space is slightly more general than Mackey's definition [7], in which it is demanded that the structure be a σ-field rather than a σ-ring; as it turns out, most of our theorems and examples refer to the more restricted kind of space anyway. In [5] and [2], the word "measurable" is used in the sense that "Borel" is used here.

[4] A *Borel mapping* is a mapping such that the inverse image of every Borel set is Borel. It is called "Borel function" in [7] and "measurable transformation" in [2, 5].