ON THE REGULARITY OF MARKOV PROCESSES[1]

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Introduction

This study is concerned with continuous parameter Markov processes having values in an arbitrary space. More specifically, we shall consider the effect of stopping times on such processes. We first define these objects following [1].

DEFINITION 1. Let X be a space, and let $\mathcal{B}$ be a σ-field of subsets of X. Let $X(t)$ be a stochastic process, defined either for $t \geqq 0$ or for $-\infty < t < \infty$ on a probability space (Ω, F, P), with X as state space and $\mathcal{B}$ as measurable field. Finally, let $\{F(t)\}$ be a family of σ-subfields of F, defined for the same range of t as $X(t)$, such that $F(t_1) \subset F(t_2)$ for $t_1 < t_2$. Then $X(t)$ is a *Markov process relative to the family* $\{F(t)\}$ if (a) for each t and $E \in \mathcal{B}$,

$$\{X(t) \in E\} \in F(t),$$

and (b) for $t_1 < t_2$ and $E \in \mathcal{B}$,

$$P(\{X(t_2) \in E\} \mid F(t_1)) = P(\{X(t_2) \in E\} \mid X(t_1)) \quad \text{a.s.}$$

(a.s. abbreviates "almost surely" or "with P-measure 1").

DEFINITION 2. A random variable T on Ω with values in

$$\{\infty\} \cup \{t: X(t) \text{ is defined}\}$$

is a *stopping time in the general sense* for $X(t)$ if $P\{T < \infty\} > 0$ and for each t one has $\{T < t\} \in F(t)$. A *stopping time in the narrow sense* is defined by replacing $\{T < t\}$ by $\{T \leqq t\}$ in the above.

We note first that every stopping time in the narrow sense is also a stopping time in the general sense. Henceforward, *stopping time* will be used to mean stopping time in the general sense.

Along with any stopping time T for $X(t)$ we consider the new probability space derived from (Ω, F, P) by restriction to the set $\{T < \infty\}$.

DEFINITION 3. Let $X_T(t)$ be the process $X(t)$ restricted to $\Omega \cap \{T < \infty\}$, with field composed of the sets $S \cap \{T < \infty\}$, $S \in F$, and probability measure $P(S \cap \{T < \infty\})/P\{T < \infty\}$ for each such set. Let (Ω_T, F_T, P_T) designate this probability space, and let T_T be the restriction of T to Ω_T. For brevity we will omit the subscript in T_T.

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