FIXED-POINT THEOREMS FOR COMPACT CONVEX SETS

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1. Historical remarks

L. E. J. Brouwer [1] proved the well-known *Brouwer fixed-point theorem*. Let K be an n-cell, that is, a homeomorphic image of an n-dimensional cube. Let f be a continuous function from K into itself. Then there is a point P of K such that $f(P) = P$.

Schauder [9] extended the domain of validity of this theorem by demonstrating the *Schauder fixed-point theorem*. If K is a compact convex subset of a normed linear space X, and if f is a continuous transformation which carries K into itself, then there is at least one point P of K left fixed by f; that is, $f(P) = P$.

This was generalized next by Tyhonov [10] when he showed that Schauder's proof could be adapted to prove the existence of a fixed point even if X is a locally convex linear topological space instead of a normed space.

It is clear that if P is fixed under f, then it is also a fixed point of every power of f, $f^2 = f \circ f$, $f^3 = f \circ f^2$, and so on; that is, P is fixed under the smallest semigroup of operators on K which includes f. In the same way, P is a fixed point of every one of the functions f of a family F of functions from K into K, if and only if P is also a fixed point of every finite product, $\bigcirc_{i \leqq n} f_i$, of functions from F.

It is not known, *even for the one-dimensional case*, whether every two commuting continuous functions from K into K share a common fixed point (see the research problem of Isbell [6]). This adds to the interest and value of the generalization of a special case of Tyhonov's theorem due to Kakutani [7] and A. A. Markov [8].

KAKUTANI-MARKOV THEOREM. *Let K be a compact convex set in a locally convex linear topological space, and let F be a commuting family of continuous, affine transformations, f, of K into itself. Then there is a common fixed point of the functions in F; that is, there is an x in K such that $f(x) = x$ for every f in F.*

As in the Tyhonov theorem we observe that it costs nothing in this theorem to replace F by $\Sigma(F)$, the smallest semigroup of continuous, affine transformations of K into itself which contains F. In this case the commutativity of the family F is carried to the semigroup $\Sigma(F)$, so the theorem above is equivalent to that obtained by replacing the word "family" by "semigroup".

The property of commutativity is not shared by all semigroups. We discuss here another property which all commutative semigroups have and some other

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