VECTOR LOOPS

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In this paper we shall study commutative diassociative loops with operators from a division ring (a loop is called *diassociative* if any pair of elements generate a subgroup, or associative subloop). Since these loops offer a generalization of the concept of a vector space, we have called them vector loops. The basic result, from which everything else follows, is the setting up of a one-to-one correspondence between classes of vector loops and certain geometrical systems. This correspondence is a generalization of the usual method of coordinatizing a Desarguesian projective plane by triples of elements from a division ring.

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1. Basic definitions and terminology

We begin with a rigorous definition of a vector loop. Let V be a loop, D any division ring, and suppose that the elements of D induce endomorphisms on V satisfying the three axioms:

(a) $(\alpha x + \beta y) + (\gamma x + \delta y) = (\alpha + \gamma)x + (\beta + \delta)y$, *for all* $x, y \in V$, $\alpha, \beta, \gamma, \delta \in D$.

(b) $\alpha(\beta x) = (\alpha\beta)x$ *for all* $x \in V$, $a, \beta \in D$.

(c) $1x = x$, *where* 1 *is the identity element of* D.

Then V will be called a *vector loop* over the division ring D. As in the associative case, we shall call a subloop of V a *subspace* whenever it is closed under multiplication by D, and we shall call an element x of V a *multiple* of an element y of V if there exists a scalar α in D such that $x = \alpha y$. A set $\{x_i\}$ of elements of a vector loop V will be called a *set of representatives of* V if every nonzero element of V is a multiple of one and only one of the x_i's. Also if $\{x_i\}$ is any set of elements in V, we shall define the subspace S *generated* by the set $\{x_i\}$ to be the intersection of the subspaces containing the set (we shall usually use the symbol $\langle x_i \rangle$ to denote the subspace generated by the set $\{x_i\}$). Whenever no proper subset of the set $\{x_i\}$ generates S, we shall say that $\{x_i\}$ is a *strongly independent* (or just independent) set. If an ordered set $\{x_i\}$ of elements of V satisfies the condition that for no finite subset $x_{i_1}, \cdots, x_{i_n}$ (arranged in ascending order) of $\{x_i\}$ do there exist scalars $\alpha_1, \cdots, \alpha_n$, not all zero, such that $(\cdots((\alpha_1 x_{i_1} + \alpha_2 x_{i_2}) + \alpha_3 x_{i_3}) \cdots + \alpha_n x_{i_n}) = 0$, then

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