PICARD BUNDLES

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On a nonsingular algebraic variety V of dimension n, a large maximal family of positive $(n - 1)$-cycles is naturally fibered by the linear systems into which these divisors may be grouped. The algebraic variety parametrizing these "fibers" is the Picard variety of V; it is a complete group variety. This is the construction of the Picard variety given by Matsusaka, following ideas of Chow; it is still the only construction which gives it *a priori* as a projective variety.

The relation between these maximal families and the Picard variety may be expected to shed light on both. For example, using the results here proved, we have shown that when V is a curve C, one is led to structural information about the rational equivalence ring of high symmetric products of C as well as to certain relations in the rational equivalence ring of its Jacobian. Again, in the classical case, Kodaira [2] has in this way studied the characteristic series of maximal families of divisors, while the projective character of Matsusaka's construction holds out some hope of studying the behavior of Picard varieties under specialization. We devote therefore the first part of this paper to showing that under the obvious geometric hypotheses, a complete maximal family of positive divisors is actually an algebraic projective bundle over the Picard variety. The essential point here is to prove the algebraic local triviality; that it is a bundle follows automatically, since we are dealing with projective bundles.

Once one has a projective bundle, one significant question is whether or not it has cross-sections. We show in the second part that these indeed exist, if the family of divisors is large enough. In this way one gets algebraic families of divisors parametrizing the Picard variety in a one-one way. Such families were constructed by Weil in the classical case [9], in order to show that analytic sets of divisors were mapped analytically into the Picard variety. We follow his ideas, which are basically algebraic, only taking some care to perform the construction as efficiently as possible.

For the special case when V is a curve C, the Picard bundle is the n-fold symmetric product $C(n)$, $n > 2g - 2$, and one can then obtain from the preceding proof the explicit estimation that $C(n) \rightarrow J$ has cross-sections if $n > 4g$. Thus high symmetric products contain Jacobians as subvarieties. Just how good the estimation is seems hard to say; in truth we would be happier with $n > 3g$, but do not seem to be able to get it. In any event, we prove in the last section that at least it does not always have cross-sections for all n; namely, if $n = 2g - 1$ and $g > 1$, it has in general none. This follows from a general criterion for nonexistence of cross-sections of projective

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