# CONFORMALLY INVARIANT CLUSTER VALUE THEORY

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## 1. Introduction

The purpose of this paper is to show how potential and probability theory can be used jointly to set up a conformally invariant cluster value theory of analytic functions. The analytic functions studied are the intrinsically natural ones, those whose domains and ranges are Riemann surfaces, and the methods used would not become any simpler if the functions were meromorphic functions defined on plane domains with smooth boundaries.

The present paper does not pretend to completeness, of course. What it contains is the substructure of a theory, and the proofs of a few key theorems. It is noteworthy that the Fatou boundary limit theorem for numerically-valued functions regular on a disc, and its generalization to functions of bounded type, are valid in our general context. This makes it possible to give a simple interpretation, in terms of boundary functions, to Heins's class **Bl** of analytic functions, a generalization of Seidel's class **U** and Storvick's class **L**. As an application to a classical situation, a disc covering theorem with a hypothesis of a different type from Bloch's is proved.

The probabilistic basis of the work is the theory of Brownian motion on a Riemann surface. This provides a suitable path system on any Riemann surface, replacing, for example, the set of radii to the perimeter of a disc. The key potential-theoretic tool is the Martin boundary $R^M$ of a hyperbolic Riemann surface $R$, together with the Martin topology and the Cartan-Brelot-Naïm fine topology on $R \cup R^M$.

The reader will observe that sometimes the new theorems are not strictly generalizations of their classical versions, because the classical versions do not have an invariant form, so that the general theorems do not reduce to exactly the classical ones under the classical hypotheses. For example, the Fatou theorem that a function bounded and regular on a disc has an angular limit at almost every (Lebesgue measure) perimeter point becomes the theorem that an analytic function from a hyperbolic Riemann surface $R_1$ into a hyperbolic Riemann surface $R_2$ has a boundary limit (on $R_2 \cup R_2^M$) on approach in terms of the fine topology to almost every (harmonic measure) point of $R_1^M$. This general theorem is a perfect generalization of the classical theorem even though angular approach to a point of the perimeter of a disc is not the same as approach to the point in the fine topology. In fact if a bounded regular function on a disc has a fine limit at a perimeter point, it has this limit as a limit along certain continuous (conditional Brownian) paths, and hence has this limit also on angular approach, by a classical argument. Thus the gen-

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