COHOMOLOGY OF ALGEBRAIC LINEAR GROUPS

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1. Introduction

The theory of rational representations of algebraic linear groups over fields of characteristic 0 has, for some time, been in a sufficiently well developed state to call for an adaptation of homological algebra to the requisite category of "rational modules". The most elementary portion of this program is carried out in Section 2 below, and this sets the stage for what follows.

In the later results, a vital role is played by the decomposition of an algebraic linear group into a semidirect product of the maximum unipotent normal subgroup by a fully reducible subgroup. This was established, for not necessarily irreducible algebraic linear groups over fields of characteristic 0, by Mostow in [7]. In the irreducible case, the group decomposition follows easily from the corresponding decomposition of the Lie algebra. However, the proof of the general case seems to require Mostow's result on the conjugacy of the maximal fully reducible subgroups. For this reason, and also by way of illustration, we apply (in Section 3) the elementary theory of rational modules to obtain a simple direct proof of the conjugacy theorem. At the same time, we sketch the resulting simplification in the proof of the decomposition theorem, and we discuss the decomposition with reference to representations and group extensions.

Sectons 4 and 5 contain the main results. The suggestion for these comes from the results of van Est [9] on the differentiable cohomology of Lie groups. They concern the relations between the rational group cohomology, the ordinary Lie algebra cohomology, and the cohomology of the differential forms. It is due to the semidirect product decomposition that the results for algebraic linear groups are more precise than van Est's results for Lie groups, which concern a situation that is somewhat more general than the straight analogue of what is considered here.

Van Est's theory has been rounded out and strengthened by Mostow (in [8]), and it has become clear from Mostow's approach that a cohomology theory of groups that takes account of additional structure (topological, differentiable, or algebraic) must be based on *injective* resolutions in the requisite category of modules while, contrary to the case of discrete groups, the *projective* part of the machinery of homological algebra is inapplicable. This realization was the point of departure for the present investigation.

In Section 6, we apply the results on the rational group cohomology to obtain the expected interpretation of the 2-dimensional rational cohomology groups as groups of equivalence classes of rational group extensions. This

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