ON THE INTERPOLATION OF L^p FUNCTIONS BY JACKSON POLYNOMIALS

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1. Through the work of Marcinkiewicz and Zygmund [2], it is known that there is an analogy between the behaviour of the Fourier series and certain trigonometric polynomials corresponding to a given function f of class L^p, $p > 1$. The polynomials which they consider are of two types, the ordinary interpolating polynomials and the Jackson polynomials. However, the usual points of interpolation are translated by an arbitrary real u. Thus we write

$$(1)\qquad \begin{aligned} I_{n,u}(x;f) &= \frac{2}{2n+1}\sum_{j=0}^{2n} f\left(u+\frac{2\pi j}{2n+1}\right) D_n\left(x-u-\frac{2\pi j}{2n+1}\right), \\ J_{n,u}(x;f) &= \frac{2}{n+1}\sum_{j=0}^{n} f\left(u+\frac{2\pi j}{n+1}\right) K_n\left(x-u-\frac{2\pi j}{n+1}\right) \end{aligned}$$

for the ordinary and Jackson polynomials of order n respectively. D_n and K_n are the Dirichlet and Fejer kernels respectively. Each interpolates the periodic function f at the corresponding interpolation points. The Jackson polynomials have a certain smoothness, and the fact that the kernel is positive makes them easier to treat. Since our own results are much more precise in this case, we emphasize their treatment and reduce to the status of corollaries our results about the sequence $I_{n,u}(x;f)$.

It is known [1] that for every $p > 0$ there are functions f in L^p such that the sequence $J_{n,u}(x;f)$ diverges for almost every (x, u): i.e., x and u are real variables, and the exceptional set is of two-dimensional zero measure. In the next section, we make this result exact by proving an order condition for Jackson polynomials which is shown to be best possible by examples. The construction of the examples involves the sharpening of a known technique [1]. In the following section, the positive result is generalized to certain sequences of linear operators; and this general result is then applied to the ordinary interpolating polynomials $I_{n,u}(x;f)$. Finally we apply our general result to the case of Riemann sums; and from this follows a result on localization theory for Jackson polynomials.

2. THEOREM 1 (i). *Let $p > 1$. Given f in L^p, then for almost every (x, u)*

$$\lim_n |J_{n,u}(x;f)|^p/n = 0.$$

(ii). *Given any positive sequence $\omega(n) = o(n)$ and $p > 0$, there exists a function f of class L^p such that for almost every (x, u)*

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