FUNCTIONALS RELATED TO MIXED VOLUMES

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We denote by R^n a fixed euclidean space of dimension n. A coset of a sub-vector-space of R^n, of dimension m, will be termed m-flat. $\mathcal{C}$ stands for the family of all compact, *convex* subsets of R^n. An $A \in \mathcal{C}$ will be called a convex body; it will be termed proper if it has inner points in R^n. $\mathcal{C}$ is a locally compact, separable, metric space with the topology introduced by Minkowski and Blaschke. A real valued, *continuous* function $\varphi : \mathcal{C} \to R$ will be called a *functional* (of convex bodies). We will deal only with φ's having the following properties:

$$\varphi(tA) = \varphi(A) \qquad (t:R^n \to R^n;\ t(x) = x + x_0) \tag{1}$$

$$\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B) \qquad (A, B, A \cup B \in \mathcal{C}). \tag{2}$$

We choose now a proper convex body U, to be fixed in the rest of this note. Theorem 1 below could be formulated in terms of Minkowski integral geometry [3], [4], the convex body U being either the indicatrix, or the isoperimetrix. However, in the present note, we do not want to pursue this direction. It suffices to say, in order to suggest the role of U in the present context, that, if we substitute the unit ball, $B^n = \{x : x \in R^n, \| x \| \leqq 1\}$, in place of U in Theorem 1 below, the statement is a well known and useful theorem of euclidean geometry.

The mixed volumes [2; p. 40]

$$\varphi_i(A) = V_i(A, U) \qquad (i = 0, \cdots, n), \tag{3}$$

considered as functions of the first argument, are particular functionals having properties (1), (2). If $U = B^n$, φ_0, φ_1 are proportional to the volume and surface area, respectively. Furthermore, it is well known [7; p. 221], that the functionals

$$W_i(A) = V_i(A, B^n) \qquad (i = 0, \cdots, n) \tag{4}$$

form a basis in the vector space of the functionals ψ, which are additive, in the sense of (2), and are invariant under isometries, that is to say, such that

$$\psi(gA) = \psi(A) \qquad (g : R^n \to R^n;\ g \text{ isometry}) \tag{5}$$

holds true for every isometry g.

A weaker form of this statement is the following. A functional ψ is of the form

$$\psi(A) = \textstyle\sum_{i=0}^{n} \alpha_i W_i(A) \qquad (\alpha_i \in R) \tag{6}$$

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