HOMOMORPHISMS OF MEASURE ALGEBRAS

BY

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1. Introduction

In their recent paper [2] Hewitt and Kakutani prove a truly remarkable theorem: *Let G be a locally compact Abelian group, and let $M(G)$ be the measure algebra on G. Let P be an independent subset of G, and denote by $M(P \cup -P)$ the linear subspace of measures concentrated on $(P \cup -P)$. If L is any linear functional on $M(P \cup -P)$ of norm 1 and satisfying the property $L(\sigma_x)L(\sigma_{-x}) = 1$ for every $x \in P$, then there is a homomorphism h defined on all of $M(G)$ which agrees with L on $M(Q)$.*

Their proof is an existence proof. In this paper we actually *construct* such a homomorphism. This construction, we believe, contributes to a better understanding of the complexities of measure algebras. It is easy to prove, via this construction, that the extension of a linear functional to a homomorphism is *unique* if restricted to the subalgebra M defined below. In a later paper we hope to use this fact to describe the ideal space of M and to give an analysis of this subalgebra.

We outline the procedure for constructing the homomorphism. Let M_0 be the algebra generated by $M(P \cup -P)$ and all the discrete measures. Then let M_1 be the algebra consisting of all those measures which are absolutely continuous to some element of M_0. We let $h = L$ on $M(P \cup -P)$ and extend h to M_1 making use of Šreĭder's "generalized functions" (see [3]). After proving h is well defined and h is a homomorphism on M_1, we extend h to be a homomorphism on the closure M of M_1. Next, we show that the orthogonal complement $M^{\perp}$ of M is an ideal and $M(G)$ is the direct sum of M and $M^{\perp}$. We conclude by defining $h(\mu) = h(\mu_M)$ where $\mu \in M(G)$ and μ_M is the projection of μ on M.

In §3 we prove a "generalized Lebesgue decomposition theorem" which plays a small but important role in our construction. In §4 we construct the homomorphism.

2. Preliminaries

Throughout this paper we assume G is a locally compact Abelian additive group. We let $M(G)$ be the set of all complex-valued regular Borel measures on G. It should be noted that Haar measure m is in $M(G)$ if and only if G is compact. With addition and scalar multiplication defined in the obvious way, $M(G)$ is a Banach space under the norm of total variation, i.e., $\| \mu \| =$

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