A GENERALIZATION OF THE RIEMANN-ROCH THEOREM

BY

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1. Introduction

In this paper[1] a Riemann-Roch theorem is proved for a module, over a function field K, which is under the action of simple algebras over K. Specialization of this module leads on one hand to the Riemann-Roch theorem of E. Witt [16] for simple algebras over K, and on the other hand to an extension of A. Weil's Riemann-Roch theorem for matrices over function fields [15], in the case that his "signature" is taken to be identically 1. In each case the constant field is allowed to be arbitrary.

There is also a brief account (in §2), partly new in method, of the arithmetic of simple algebras over K. In §3 our generalization of the Riemann-Roch theorem is proved for a certain module over the function field K. In §4 this module is taken to be a simple algebra A over K; a restriction of the definition of divisor then leads to a suitably specific form of the Riemann-Roch theorem for A. Related questions—the different, the Riemann-Hurwitz formula, and a genus-like invariant of A—are then discussed. Finally, in §5, it is shown that our Riemann-Roch theorem for A implies that of Witt [16]. The paper concludes with a theorem extending the generalized Riemann-Roch theorem of Weil [15] (when his "signature" is trivial) for matrices over function fields.

Part of the origin of this kind of investigation is in the papers of Hecke [7, 8], Chevalley and Weil [3], and Weil [14], which are concerned with the problem of decomposing into its irreducible parts a certain natural representation of $G/H(N)$, where G is the modular group and $H(N)$ the subgroup of matrices congruent (mod N) to the identity, as linear transformations of the space of "cusp forms" of type $(2, N)$. Since there is a natural isomorphism between this space of cusp forms and the differentials of the first kind of the associated function field $K_{H(N)}$, the problem can be transformed to one in terms of matrices over $K_{H(N)}$.

The methods used here are those of linear topology and duality, first applied to this kind of problem by K. Iwasawa in [10] and particularly [9]. The proofs in §3 are direct generalizations of the proofs of Iwasawa for the corresponding theorems about K. Indeed, much of this paper may be thought of as the tensor product of the appropriate spaces over K with [9].

I wish to thank Professor Iwasawa for suggesting to me the problems dealt with here. As I have indicated, his works [9] and [10] made these problems

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