ON A THEOREM OF SPITZER AND STONE AND RANDOM WALKS WITH ABSORBING BARRIERS

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1. Introduction

Consider a sequence $X_1, X_2, \cdots$ of independent, identically distributed random variables, taking integer values only. We assume that every integer is a possible value (compare [5]), i.e., if

$$S_n = \sum_{i=1}^{n} X_i, \tag{1.1}$$

then there exist integers u and v such that[1]

$$P\{S_u = +1\} > 0 \quad \text{and} \quad P\{S_v = -1\} > 0. \tag{1.2}$$

Let I be any finite set of integers, containing $\mu(I)$ points, and put

(1.3) $N_I(A)$ = the number of terms S_k in the infinite sequence $S_1, S_2, \cdots$, such that $S_k \in I$ and $S_i \leqq A$ for $i = 1, 2, \cdots, k$,

(1.4) $N_I(A, -B)$ = the number of terms S_k in the infinite sequence $S_1, S_2, \cdots$ such that $S_k \in I$ and $-B \leqq S_i \leqq A$ for $i = 1, 2, \cdots, k$.

In a recent research note (Theorem 6 of [11]) of Spitzer and a paper [13] of Spitzer and Stone the asymptotic distributions of $N_I(A)$ and $N_I(A, -A)$ were given for the case $\mu(I) = 1$, X_i symmetrically distributed and $EX_i^2 < \infty$. At the same time Spitzer suggested in [11] that some formulae would be valid for any finite $\mu(I)$ and even for nonsymmetrically distributed X_i with zero mean. We shall drop the condition $EX_i^2 < \infty$ but instead assume that the characteristic function $\phi(t) = Ee^{itX_1}$ is such that

$$\lim_{t \downarrow 0} (1 - \phi(t))/t^{\alpha} = Q \qquad \text{with} \quad \operatorname{Re} Q > 0$$

for some α with $1 \leqq \alpha \leqq 2$. (In some places $0 < \alpha < 1$ is also considered.)

The generalizations suggested by Spitzer for $N_I(A)$ will be derived, and the corresponding results for $1 \leqq \alpha < 2$ are also found. If there are two barriers, we consider mostly variables with symmetric dlstributions, i.e., for which $P\{X_i = k\} = P\{X_i = -k\}$. We do not require, however, that the barriers be symmetrically placed, i.e., we shall find the asymptotic distribution of $N_I(A, -B)$ where B not necessarily equals A.

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[1] As usual, $P\{A\}$ = probability of the event A;
$P\{A \mid B\}$ = conditional probability of A, given B;
$E\{X\}$ = expectation of the random variable X; and
$E\{X \mid B\}$ = conditional expectation of X, given B.