GENERALIZATIONS OF A THEOREM OF N. BLACKBURN ON p-GROUPS

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Let G be a finite p-group, and let $G = G_1 \supset G_2 \supset \cdots$ be the descending central series of G. N. Blackburn [1] has shown that if G_2 can be generated by two elements, then G_2 is a metacyclic group of class at most two. We generalize this theorem in two ways. We first show (Theorem 1) that Blackburn's theorem holds if "G_2" is replaced by $\Psi(G)$ where $\Psi(G)$ is any one of a large class of characteristic subgroups of G. Secondly, we show (Theorem 2) that if G_n can be generated by n elements then the Frattini subgroup of G_n coincides with the subgroup generated by the p^{th} powers of elements of G_n. If p is odd, this result for $n = 2$ is equivalent to Blackburn's theorem. An application of Theorem 2 to the problem of bounding the length of the derived series of a p-group is given in Remark 2.

All groups considered are finite p-groups. We use the following notation: $P(G)$ is the subgroup generated by the p^{th} powers of elements of G; $\Phi(G)$ is the Frattini subgroup of G; $G^{(k)}$ is the k^{th} derived group of G; $G = G_1 \supseteq G_2 \supseteq \cdots$ is the descending central series of G; $1 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots$ is the ascending central series of G; $|G|$ is the order of G; $(h, k) = h^{-1}k^{-1}hk$; (H, K) is the subgroup generated by the set of all (h, k) for $h \in H$ and $k \in K$. The group G is said to be metacyclic if G contains a cyclic normal subgroup N such that G/N is cyclic.

We denote by Ψ a rule which assigns a unique subgroup $\Psi(G)$ to every p-group G. We consider only those rules for which

(1) $\Psi(G)$ is a characteristic subgroup of G,

(2) $\Psi(G) \subseteq \Phi(G)$, and

(3) $\Psi(G/N) = \Psi(G)/N$ whenever $N \subseteq \Psi(G)$ and N is normal in G.

For example, one could let $\Psi(G) = G_n$ for any $n \geqq 2$.

We shall need two lemmas.

LEMMA 1. *If* $|N| \leqq p^n$ *and* N *is normal in* G, *then* $N \subseteq Z_n$.

Proof. This result is well known for $n = 1$, and the general case follows by an easy induction.

The next lemma follows from the fact that the automorphism group of a cyclic group is abelian.

LEMMA 2. *Every cyclic normal subgroup of* G *is centralized by* $G^{(1)}$.

THEOREM 1. *If* $\Psi(G)$ *can be generated by two elements, then* $\Psi(G)$ *is meta-*

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