ONE-DIMENSIONAL TOPOLOGICAL SEMILATTICES

BY

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1. Introduction

In [5] A. D. Wallace proved that a compact, connected mob with zero and unit has trivial cohomology groups for $n > 0$. It is implicit in this result that if such a mob is one-dimensional[2] and locally connected, then it is a tree. For, if X is a continuum, $\dim X = 1$, and $H^1(X) = 0$, then X is hereditarily unicoherent; thus, if X is locally connected, it is a tree [8]. In the main theorem of this note we modify Wallace's result so as to eliminate the necessity of hypothesizing a unit. Specifically, we prove

THEOREM. *A compact, connected, locally connected, one-dimensional, idempotent, commutative mob is a tree.*

2. Preliminaries

A *topological semilattice* (= TSL) is an idempotent commutative mob. A TSL can be endowed with a natural partial ordering by letting $x \leqq y$ if $xy = x$. Thus $xy = \text{g.l.b.}\ (x, y)$, denoted hereafter by $x \wedge y$, and this partial ordering is continuous in the sense that its graph $(= \{(x, y) : x \leqq y\})$ is closed. It is easy to see that a compact TSL is $\wedge$-complete and therefore has a zero. Also a $\wedge$-complete TSL with unit is an algebraic lattice (but not necessarily topological).

A *tree* is a continuum (= compact connected Hausdorff space) in which every two points are separated by a third point. A tree admits a partial ordering as follows: Select a point x_0, and define $x \leqq y$ if and only if $x = x_0$, or $x = y$, or x separates x_0 and y. This partial ordering is called the *cutpoint ordering* of a tree [6]. We recall [7] that a compact Hausdorff space X is a tree if, and only if, X admits a partial ordering, $\leqq$, such that for each $a, b \in X$

(i) $L(a)$ and $M(a)$ are closed,[3]

(ii) if $a < b$, then there exists $c \in X$ with $a < c < b$,

(*) (iii) $L(a) \cap L(b)$ is a nonvoid chain,

(iv) $M(a) - \{a\}$ is open.

3. Proof of the theorem

Throughout this section, S will denote a compact, connected, locally con-

Received July 31, 1959; received in revised form July 5, 1960.

[1] This research was supported in part by the Air Force Office of Scientific Research.

[2] The dimension function employed throughout this note is *codimension* as expounded by Haskell Cohen [3]. For a compact Hausdorff space, the codimension (with the integers as coefficient group) and the covering dimension agree.

[3] In a partially ordered set we write $L(a) = \{x : x \leqq a\}$ and $M(a) = \{x : a \leqq x\}$.