# HOMOTOPICAL NILPOTENCY

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## Introduction

Let $X$ be a topological space with base-point, $\Omega X$ its loop space, $\Sigma X$ its (reduced) suspension. The ordinary multiplication and inversion of loops convert $\Omega X$ into an $H$-space. Eckmann and Hilton [6] have shown that, dually, the identification map resulting by pinching to a point the equatorial $X \subset \Sigma X$ and the reflection of $\Sigma X$ in $X$ may be used to convert the suspension into an $H'$-space, the dual of an $H$-space. Just as in group theory, for every $n \geqq 1$ a commutator map of weight $n$ is available in any $H$-space; accordingly, we define the nilpotency class of an $H$-space as the least integer $n \geqq 0$ (if any) with the property that the commutator map of weight $n + 1$ is nullhomotopic. The concepts of a commutator map and of nilpotency class may readily be dualized to $H'$-spaces: for every $n \geqq 1$ there results a co-commutator map of weight $n$ and the co-nilpotency class of an $H'$-space is the least integer $n \geqq 0$ (if any) with the property that the co-commutator map of weight $n + 1$ is nullhomotopic. We now revert to the topological space $X$ and introduce two integers, which may be finite or not: the nilpotency class $\operatorname{nil} \Omega X$ and the co-nilpotency class $\operatorname{conil} \Sigma X$. They are uniquely determined by the based homotopy type of $X$.

The paper is divided into six parts. The first contains basic definitions concerning $H$- and $H'$-spaces, commutator and co-commutator maps, nilpotency and co-nilpotency classes. In the second part, we present results relating the nilpotency and co-nilpotency classes of $H$- and $H'$-spaces to the nilpotency class of certain groups of homotopy classes of maps; some of these results provide further motivation for our concept of co-nilpotency of an $H'$-space.

Given a base-points-preserving map $f: X \to Y$, the nilpotency class $\operatorname{nil} \Omega f$ is the least integer $n \geqq 0$ for which the composition

$$\Omega X \times \cdots \times \Omega X \xrightarrow{\varphi_{n+1}} \Omega X \xrightarrow{\Omega f} \Omega Y$$

is nullhomotopic; here, $\varphi_{n+1}$ is the commutator map of weight $n + 1$, and $\Omega f$ is induced by $f$ in the obvious way. In the third section we prove that if $\eta: Q \to Y$ is the inclusion map of the fibre $Q$ in the total space $Y$, then $\operatorname{nil} \Omega Q \leqq 1 + \operatorname{nil} \Omega \eta$. Dually, if $\eta: X \to P$ is the projection of $X$ onto the "cofibre" $P$, i.e., $\eta$ is the identification map resulting by pinching to a point a subset which is smoothly imbedded in $X$, then $\operatorname{conil} \Sigma P \leqq 1 + \operatorname{conil} \Sigma \eta$; the definition we give of $\operatorname{conil} \Sigma \eta$ stands in evident duality to that of $\operatorname{nil} \Omega \eta$. In particular, $\operatorname{nil} \Omega Q \leqq 1 + \operatorname{nil} \Omega Y$ and $\operatorname{conil} \Sigma P \leqq 1 + \operatorname{conil} \Sigma X$. The first

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