TORSION-FREE RINGS

BY

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1. Introduction

In the following we are concerned with associative rings which have a torsion-free abelian group as additive group. Such rings are called *torsion-free rings*. The rank of a torsion-free ring is the rank of its additive group, which is the cardinality of a maximal independent set of elements.

The tensor product $R \otimes A$ [6], where R is the field of rational numbers and A is a torsion-free ring, can be made into an associative algebra over R by defining $r(s \otimes a) = rs \otimes a$ for $r, s \in R$, $a \in A$. We will denote $R \otimes A$ by A^*. It can be readily verified that A^* has the following properties:

(1) Every element of A^* can be written in the form $r \otimes a$, $r \in R$, $a \in A$.

(2) A is imbedded as a subring in A^*. Since A is torsion-free, the mapping $a \to 1 \otimes a$ is an imbedding [6, p. 130].

(3) For every $\bar{a} \in A^*$, there exists an integer $n \geqq 1$ such that $n\bar{a} \in A$. This follows from (1) and (2).

(4) The dimension of A^* over R is equal to the rank of A.

(5) A^* is a unique smallest associative algebra over R containing A as a subring.

Because of (1), we simplify the notation by writing ra instead of $r \otimes a$ for the elements of A^*. It should be noted that as an additive group A^* is just the minimal divisible torsion-free group containing A [9, p. 66], and that when A^* is regarded as the set of formal products ra, $r \in R$, $a \in A$, certain identifications which we make in the sequel are clear. We introduce the following terminology.

Definition 1.1. *The algebra $A^* = R \otimes A$ is called the algebra type of the ring A, and torsion-free rings A_1 and A_2 are said to have the same algebra type if their algebra types A_1^* and A_2^* are isomorphic algebras.*

Definition 1.2. *Let G be a torsion-free abelian group, and let T be an associative algebra over R. Then G admits a multiplication of algebra type T if there exists a ring A with additive group A^+ isomorphic to G such that A^* and T are isomorphic algebras.*

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