FOURIER SERIES OF AUTOMORPHIC FORMS OF NONNEGATIVE DIMENSION[1]

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I. Introduction

In [2] Rademacher introduced a method for recapturing the functional equation $J(-1/\tau) = J(\tau)$ directly from the Fourier series expansion for $J(\tau)$, the well-known modular invariant. In this paper we extend his method to entire forms of positive even integral dimension for the modular group and for the groups $G(\sqrt{2})$, $G(\sqrt{3})$, $G(2)$, where $G(\sqrt{l})$ is generated by $\tau' = \tau + l^{1/2}$, $\tau' = -1/\tau$ $(l = 2, 3)$, and $G(2)$ is the principal congruence subgroup of level two, of the modular group. This group is generated by $\tau' = \tau + 2$, $\tau' = \tau/(2\tau + 1)$.

We start here with the Fourier series given by Rademacher and Zuckerman [3], Raleigh [4], and Simons [6]. In [3] Fourier expansions are given for entire modular forms (i.e., modular forms regular in the upper half plane) of positive dimension. While every entire modular form of positive dimension has a Fourier series of the type given in [3], the converse is not true. That is, not every function defined by such a Fourier series is a modular form. However, it is reasonable to expect "decent" behavior under modular substitutions for all such Fourier series. We show that this is indeed true in a certain special case described below. Using these results it is a simple matter to construct modular forms of positive even integral dimension by means of their Fourier series.

Similar results will then be obtained for the groups $G(\sqrt{2})$, $G(\sqrt{3})$, and $G(2)$.

The result of Rademacher and Zuckerman [3] is as follows.

THEOREM (1.01). *Let $F(\tau)$ be a modular form of dimension $r > 0$; that is,*

(a) $F((a\tau + b)/(c\tau + d)) = \varepsilon(a, b, c, d) \cdot (-i(c\tau + d))^{-r} \cdot F(\tau)$, *where a, b, c, d are integers with $ad - bc = 1$ and $c > 0$, ε does not depend on τ, $|\varepsilon| = 1$, and we choose $|\arg(-i(c\tau + d))| < \pi/2$;*

(b) $F(\tau + 1) = \varepsilon(1, 1, 0, 1)(-i)^{-r} \cdot F(\tau) = e^{2\pi i\alpha}F(\tau)$, $0 \leqq \alpha < 1$;

(c) *the Fourier expansion of* $\exp(-2\pi i\alpha\tau) \cdot F(\tau)$ *contains only a finite number of terms with negative exponents.*

Received July 15, 1959.

[1] Doctoral Dissertation, University of Illinois, 1958. This research was supported by a grant from the United States Office of Naval Research. I would like to express my thanks to Professor Paul T. Bateman for his help and encouragement throughout the preparation of this thesis. His many comments and observations have helped to improve this paper immeasurably.