# ON LAST EXIT TIMES

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**1.** The terminology and notation of this paper follow that of my book [1], where not explicitly explained. Results cited without amplification can also be found in the book.

Let $\{x_t, t \geqq 0\}$ be a well-separable, measurable Markov chain with the discrete state space $I$, the initial distribution $\{p_i\}$ and stationary, standard transition matrix $((p_{ij}))$, $i, j \in I$. Let

$$(1) \quad {}_ip_{kj}(t) = P\{x(t_0 + t, w) = j, x(t_0 + s, w) \neq i, 0 < s < t \mid x(t_0, w) = k\};$$

for every $t_0 \geqq 0$ for which the conditional probability is defined; thus ${}_ip_{kj}(t) \equiv 0$ if $k = i$ or $i = j$, by stochastic continuity. We note that if $k$ is a stable state and $k = i$, the definition (1) differs from the one adopted in [1].

Writing as usual

$$S_i(w) = \{t : x(t, w) = i\}, \qquad \overline{S_i(w)} = \text{closure of } S_i(w),$$

we define

$$(2) \qquad \gamma_i(t, w) = \sup \{\overline{S_i(w)} \cap [0, t]\}$$

and call it *the last exit time from i before time t.* The separability and measurability of the process ensure that the corresponding $w$-function $\gamma_i(t)$ is a random variable. Under the hypothesis that $x(0, w) = i$, the stochastic continuity of the process implies that $\gamma_i(t)$ has a distribution function $\Gamma_i(\,\cdot\,, t)$ vanishing at zero, continuous in $(0, t)$, and making a jump of magnitude $p_{ii}(t)$ at $t$ to reach the value one. We have clearly, if $0 \leqq s \leqq t$,

$$(3) \qquad \Gamma_i(s, t) = \sum\nolimits_{k \neq i} p_{ik}(s)[1 - F_{ki}(t - s)],$$

where $F_{ki}$ is the *first entrance time distribution from k to i.* We define similarly

$$(4) \qquad \begin{aligned} \Gamma_{ij}(s, t) &= P\{\gamma_i(t, w) \leqq s; x(t, w) = j \mid x(0, w) = i\} \\ &= \sum\nolimits_k p_{ik}(s)\ {}_ip_{kj}(t - s), \end{aligned}$$

noting that the term corresponding to $k = i$ vanishes. Thus we have

$$\Gamma_i(s, t) = \sum\nolimits_{j \neq i} \Gamma_{ij}(s, t).$$

**2.** The set of sample functions with $x(0, w) = i$ and $x(t, w) = j$ can be decomposed into subsets according to the location of $\gamma_i(t, w)$ in a dyadic partition of $[0, t]$. Since the terminating dyadics $\{v2^{-n}\}$ form a separability set,

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