ADEQUATE SUBCATEGORIES

BY

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Introduction

This paper introduces and studies the notion of a *left adequate subcategory* of an arbitrary category (and the dual notion). Definition follows.

Let $\mathfrak{A}$ be a full subcategory of $\mathfrak{C}$. For any object X of $\mathfrak{C}$, let Map $(\mathfrak{A}, X)$ denote the contravariant functor on $\mathfrak{A}$ into the category $\mathfrak{U}$ of all sets and all functions which takes each object A of $\mathfrak{A}$ to the set Map (A, X), and each mapping $f: A' \to A$ in $\mathfrak{A}$ to the function from Map (A, X) to Map (A', X) defined by $[\text{Map}\ (\mathfrak{A}, X)(f)](g) = gf$. Observe that every mapping $h: X \to Y$ in $\mathfrak{C}$ induces a natural transformation from Map $(\mathfrak{A}, X)$ to Map $(\mathfrak{A}, Y)$ by multiplication. We call $\mathfrak{A}$ *left adequate* if every natural transformation between these functors is induced by a mapping in $\mathfrak{C}$ and distinct mappings induce distinct natural transformations. *Right adequate* is defined dually.

A little thought will show that the phrase "$\mathfrak{A}$ is right adequate in $\mathfrak{C}$" is a natural formulation of the somewhat variable idea "every object of $\mathfrak{C}$ has sufficiently many mappings into objects of $\mathfrak{A}$". Then the main results of this paper are the examples. Using the usual mappings (for topological spaces, the continuous functions, and similarly for other objects), we have the following. In a category of algebras with n-ary operations, the free algebra on n generators is left adequate. In compact spaces, the 2-cell is right adequate. In sets, a single point is left adequate. The duals of these examples are less neat. For sets, a countably infinite set is right adequate if and only if no measurable cardinals exist. For compact spaces, no set of them is left adequate. The 1-cell is left adequate for Peano spaces, and for all products of Peano spaces up to the first weakly inaccessible cardinal. No nontrivial instance is found of a single algebra being right adequate for a large class, excepting the few which come from the examples mentioned by duality.

The first part of the paper is devoted mainly to inverting the notions of adequacy in order to obtain a reasonable closure operation on arbitrary categories. Left or right adequacy alone is unsuitable because a left adequate subcategory of a left adequate subcategory is not generally left adequate. Further, if $\mathfrak{A}$ is a left adequate subcategory of $\mathfrak{C}$, almost no useful restriction on $\mathfrak{C}$ can be inferred from restrictions on $\mathfrak{A}$. Neither of these objections applies to the notion of left *and* right adequacy. This does yield a closure operation. The main drawback, as the examples show, is that the operation is rather feeble.

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