FUNCTIONS WHOSE PARTIAL DERIVATIVES ARE MEASURES

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Let f denote a real-valued function on euclidean N-space such that the gradient grad f in the sense of Schwartz distribution theory is a vector-valued measure [21, p. 37]. In distribution theory functions differing in zero Lebesgue N-measure are equivalent. We intend to show that, at least in two important cases, there is a function $\bar{f}$ equivalent to f which is determined more precisely than f and is in a natural sense nicer. The function $\bar{f}$ is defined as the limit in a suitable sense of a sequence of "elementary" functions. It turns out that $\bar{f}$ is determined up to Hausdorff $(N - 1)$-measure 0.

In Part I this is done when grad f is itself a function, which is the same as to say that f is equivalent to a function locally absolutely continuous in G. C. Evans's sense. Part II concerns the more difficult case when f is an integer-valued function, dual to a current c of dimension N such that both c and its boundary bc have finite mass. In Part III we apply these results to the study of sets with finite perimeter in the sense of Caccioppoli and De Giorgi. In the special case when the boundary of a set E is a compact $(N - 1)$-manifold X such that X occupies zero Lebesgue N-measure, the perimeter of E is shown to agree with the integralgeometric $(N - 1)$-area of the inclusion map i_X.

PART I

1. Introduction

We adopt the following notation throughout: $x = (x^1, \cdots, x^N)$ is a generic point of euclidean N-space R^N $(N \geqq 2)$. For $k \leqq N$ let m_k denote Hausdorff k-dimensional measure in R^N (= Lebesgue measure for $k = N$). We use fr E, cl E for the frontier, closure of a set E, respectively. For $s > 0$, $[E]_s$ will denote the s-neighborhood of E. We write spt f for the support of f. $\mathfrak{D}$ is Schwartz's space of all infinitely differentiable functions $f(x)$ with compact support. Let BV stand for the space of all locally summable functions f such that, for each $i = 1, \cdots, N$, the i^{th} distribution theory partial derivative of f is a measure μ_i. The notation is justified by the result proved independently by Krickeberg [17] and Federer [Bull. Amer. Math. Soc., vol. 60 (1954), Abstract no. 407, p. 339] that $f \in BV$ if and only if f is locally of bounded variation in the sense of Cesari and Tonelli. We write grad f for the vector-valued measure $(\mu_1, \cdots, \mu_N)$. The total variation measure of a real- or vector-valued measure μ will be denoted by $|\mu|$; see, for ex-

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