CATEGORY AND GENERALIZED HOPF INVARIANTS

BY

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1. Introduction

The Lusternik-Schnirelmann category of a topological space X is usually defined as follows: cat $X \leqq n$ if X may be covered by n open sets each of which is contractible in X. The general reference for the properties of this homotopy invariant is [4]. It was observed by G. W. Whitehead in [9] that, for a certain class of spaces including polyhedra, this definition is equivalent to the one given below (Definition 2.1), which we make the starting point of our investigation.

If we attach a cone CA to X by means of a map $f: A \to X$, it is trivial to verify with the original definition of category—and easy to verify with ours—that cat $Y \leqq$ cat $X + 1$, where $Y = X \cup_f CA$. Our interest centers in the problem of establishing conditions under which, in fact, cat $Y \leqq$ cat X. We are motivated partly by the wish to compute the category of a 1-connected polyhedron as a function of the terms in a homology decomposition (see [3]) and partly by the observation that, in the important case $n = 2$, our problem dualizes, in the sense of [2], a familiar problem of homotopy theory. Namely, if X admits a multiplication and Y is the fibre space over X induced by a map $f: X \to A$, under what circumstances does Y admit a multiplication. Answers to this question, under certain restrictions, have been given by Copeland [1] and others in terms of the concept of *primitivity* of cohomology classes. As expected, our solution of the dual problem is primarily in terms of a concept of primitivity which we introduce for homotopy classes. Moreover if we specialize A to be a Moore space $K'(G, m - 1)$, we get fairly complete results in which the primitivity property of the map f turns out to be equivalent to the vanishing of a generalized Hopf invariant which we define for elements of homotopy groups (with coefficients) of spaces of specified category. Just as in the dual situation, if X is a suspension space, all suspension elements of $\pi_{m-1}(G; X)$ are primitive, but the converse is false. We are thus enabled to construct spaces of category 2 which are not equivalent to suspensions, answering a question first raised by T. Ganea.

The definition of category which we give suggests a related notion of *weak category* (Definition 2.2) which is a weaker hypothesis on a space X in that w cat $X \leqq n$ if cat $X \leqq n$, but the converse is, in general, false. Nevertheless certain well-known properties[1] of category generalize to weak category, includ-

Received July 13, 1959.

[1] The Whitehead theorem (see [9]) on the nilpotency class of $\pi(X, Y)$ where Y is a group-like space also generalizes to spaces X of weak category n. See a forthcoming paper by I. Berstein and T. Ganea.