ON A SPECTRUM OF SET THEORIES

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Montague essentially proved (see [8]) that the set theory ZF (Zermelo-Fraenkel) cannot be obtained from the set theory S (general set theory) by the addition of a finite number of axioms, and even the stronger fact that ZF is what we shall call later essentially reflexive over S. In [5] this was extended to the hierarchy S, ZF, ZM, ZM_2, $\cdots$, ZM_Λ, $ZM_{\Lambda+1}$, $\cdots$ of set theories given there, in which each theory is essentially reflexive over all the preceding ones. In this paper it will be shown that we can define between two theories of this hierarchy various infinite ordered sets of axiomatic set theories such that each of them is reflexive over all the preceding ones. We shall actually carry out the construction only between the theories ZF and ZM. Similar spectra of theories can be defined between any two consecutive members of the above-mentioned hierarchy. Our proof of essential reflexivity is based on a generalization of the theorem of [4].

We shall use the terminology of [13]. A set of symbols of the first order predicate calculus with equality with the usual formation rules for terms and formulae, as laid out in [13, pp. 6–7], will be called a standard formal language. If R is a standard formal language and P an unary predicate which is not a symbol of R, we construct a new formal language $R^{(P)}$ by the relativization of quantifiers in R to P (see [13, p. 24]). If A is a sentence of R, we denote by $A^{(P)}$ the sentence obtained from A by relativizing the quantifiers in it to P. Let Q be a theory with standard formalization. We define the notion of an interpretation of R in Q as in [13, pp. 20–21, 29]. This interpretation is obtained by a new theory with standard formalization Q(R) which is formalized in the language which consists of the symbols of both Q and $R^{(P)}$. The valid sentences of Q(R) are exactly those which are derivable from the set which consists of the valid sentences of Q and possible definitions of the nonlogical constants of $R^{(P)}$ in Q. Given a particular interpretation of R in Q we can define in the language R a theory with standard formalization[2] Q/R (which may be called the theory induced by Q in R) by the following stipulation: A sentence A of R is valid in Q/R if and only if $A^{(P)}$ is valid in Q(R). Let T be a given theory with standard formalization with a given interpretation in Q (in the sense of [13]). We can again define the theory Q/T. We observe that Q/T is an extension of T without new symbols.

Given the interpretation (or model) of R in Q, Q/R consists of all the

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[2] Q/R is a theory with standard formalization if $(\exists x)P(x)$ is provable in Q(R). We shall assume it tacitly whenever we deal with interpretations.