STRUCTURE OF CLEFT RINGS II

BY

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I. INTRODUCTION AND PRELIMINARIES

1A. Introduction

Let R be a ring with the minimum condition on its set of left ideals. A cleaving for R is a direct decomposition, as an additive group,

$$R = S \oplus N$$

where S is a semisimple subring and N is the radical of R. Any algebra over a field K such that R/N is a separable algebra of finite rank over K affords an example of such a ring by virtue of the Wedderburn Principal Theorem.

This paper is a sequel to [8] appearing in this journal. Here we develop the concepts of structural modules, structures of modules, and structures of rings which were introduced in [8]. Certain relations between structural modules and the lattices of submodules of a module are developed in Part II with the view of application in Parts III and IV. In Part III, particular submodules of a structural module are identified as modules which are isomorphic to those formed by the endomorphism fields of an irreducible R-module in one case and to the cohomology modules $H^1(R, \operatorname{Hom}_K(F_i, F_j))$ in another case.

The structures of rings were used in [8] to give conditions which characterized when there exists an extension $I : R \to R'$ of an isomorphism $I_0 : S \to S'$ of the semisimple components of two cleft rings R and R'. Such a condition was expressed in terms of the conformality of the structures of R and R'. In Part III, we give a condition which is equivalent to comformality, but which is simpler in statement. This condition demands that there exist an isomorphism of the structural modules which satisfies a certain commutativity relation with the coboundary operator.

In the final part, there is presented an application of these results to graded rings. A *grading* of a cleft ring R is a direct decomposition

$$R = S \oplus M \oplus M^2 \oplus \cdots \oplus M^r$$

where S is a semisimple subring, M is an (S, S)-submodule, M^q is the (S, S)-module generated by products of q elements of M and $N = \oplus_{q=1}^{r} M^q$. Here we show that there exists an extension to an automorphism of R of any isomorphisms of the semisimple component of one grading to the semisimple component of a second grading; moreover, the automorphism may be specified

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