A DIMENSION THEOREM FOR SAMPLE FUNCTIONS OF STABLE PROCESSES

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1. Introduction

The main theorem of this paper concerns the Hausdorff-Besicovitch dimension of the range of the sample functions of a stable process in R_N. Results of this sort for the symmetric stable processes were obtained earlier by McKean [6], [7] and by us [1]. The symmetric stable processes are subordinate to Brownian motion, a fact that we found useful in [1]; but there seems to be no similar relationship for the general stable processes, so a different approach is necessary.

2. Preliminaries

If F is a stable probability distribution on R_N and φ is its N-dimensional characteristic function, then either F is a (possibly degenerate) N-dimensional normal distribution, or else

$$\log \varphi(y) = i(a, y) - \lambda \mid y \mid^{\alpha} \int_{S_N} w_\alpha(y, \theta)\mu(d\theta) \tag{1}$$

for some a in R_N, $\lambda > 0$, $0 < \alpha < 2$, μ a probability measure on the surface of the unit sphere S_N in R_N. In this formula θ denotes a variable point on S_N, and the function w_α is defined by

$$w_\alpha(y, \theta) = [1 - i \operatorname{sgn}(y/\mid y \mid, \theta) \tan \tfrac{1}{2}\pi\alpha]\cdot \mid (y/\mid y \mid, \theta) \mid^{\alpha}$$

if $\alpha \neq 1$, and

$$w_1(y, \theta) = \mid (y/\mid y \mid, \theta) \mid + (2i/\pi)(y/\mid y \mid, \theta) \log \mid (y, \theta) \mid.$$

The correct interpretation of this if $y = 0$ or if $(y, \theta) = 0$ is obvious. The number α is called the index of the stable distribution. Formula (1) is due to Lévy [5]. If $\alpha < 2$, then φ is integrable, so any stable distribution of index $\alpha < 2$ has a bounded continuous density. From now on we will consider only the nonnormal stable distributions.

If F is stable of index α, then for every $k > 0$

$$F(\{x: \mid x \mid > r\})/F(\{x: \mid x \mid > kr\}) \to k^{\alpha} \quad \text{as} \quad r \to \infty.$$

This is a consequence of Theorem 4.2 of [8], and it implies that if $p > 0$, then

$$\int_{R_N} \mid x \mid^{p} F(dx) < \infty$$

if and only if $p < \alpha$.

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