ON THE DENSITY OF SETS OF INTEGERS POSSESSING ADDITIVE BASES

BY

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Let $K = \{k_0, k_1, k_2, \cdots\}$ be an infinite set of positive integers with $k_0 < k_1 < k_2 < \cdots$. Let S be the set of all integers which can be expressed as the sum of distinct elements of K. It is convenient to regard 0 (the empty sum) as belonging to S. In the special case where every element of S has a unique representation as the sum of distinct elements of K, Wintner [2] has called S a π-set with basis K. For example, the set of all nonnegative integers forms a π-set whose basis consists of the powers of 2.

The relationship between a π-set and its basis can be expressed analytically by the formula

$$\sum_{n=0}^{\infty} c_n x^n = \prod_{n=0}^{\infty} (1 + x^{k_n}) \qquad (|x| < 1),$$

where c_n is the characteristic function of S, i.e., $c_n = 1$ or 0 according as n is or is not in S.

Wintner in [2] investigated the question of when a π-set has a density, i.e., when

$$\lim_{n\to\infty} (c_0 + c_1 + \cdots + c_n)/n = \theta \tag{1}$$

exists. He proved that if

$$\lim_{n\to\infty} 2^n/k_n = \theta \tag{2}$$

exists, then S has density θ. In the present paper it is shown that (2) is necessary, as well as sufficient, for the existence of a density except possibly in the special case $\theta = 0$. Wintner's question of whether or not every π-set has a density can then easily be answered in the negative.

We remark that our methods apply to the more general case where the k_n are positive real numbers not assumed to be integers and where S is not assumed to be a π-set, provided that multiplicities are counted properly.

Theorem. *Suppose* (1) *holds with* $\theta > 0$. *Then* (2) *follows.*

Proof. Any element $m \in S$ with $m < k_n$ can only involve $k_0, k_1, \cdots, k_{n-1}$ in its representation as a sum of basis elements. There are only 2^n possible sums that can be formed from $k_0, k_1, \cdots, k_{n-1}$. Hence

$$c_0 + c_1 + \cdots + c_{k_n} \leqq 2^n + 1.$$

Dividing by k_n and letting $n \to \infty$, we find that $\liminf_{n\to\infty} 2^n/k_n \geqq \theta$. Hence, for any $\theta' < \theta$, we have $k_n \leqq 2^n/\theta' = \rho' 2^n$ for $n > n(\theta')$. Write, for

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