THE p-PERIOD OF A FINITE GROUP

BY

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If π is any finite group, the p-period of π is defined to be the least positive integer q such that the cohomology groups $\hat{H}^i(\pi, A)$ and $\hat{H}^{i+q}(\pi, A)$ have isomorphic p-primary components for all i and all A [1, Ch. XII, Ex. 11]. This is equivalent to the statement that $\hat{H}^q(\pi, Z)$ has an element of order p^ν, the highest power of p dividing the order of π [1, Ch. XII, Ex. 11].

I will say that q is a p-period for π if it is a multiple of *the* p-period. The ordinary period of the cohomology of π is, of course, the least common multiple of all the p-periods. It is known [1, Ch. XII, Ex. 11] that the p-period will be finite if and only if the p-sylow subgroup of π is either cyclic or a generalized quaternion group. The purpose of this paper is to give a simple group-theoretic interpretation of the p-period of π. The methods used here also give a cohomological generalization of Grün's second theorem [3, Ch. V, Th. 6]. This will be presented in the Appendix since it is not needed in proving Theorems 1 and 2.

THEOREM 1. *If the 2-sylow subgroup of π is cyclic, the 2-period is 2. If the 2-sylow subgroup of π is a (generalized) quaternion group, the 2-period is 4.*

THEOREM 2. *Suppose p is odd and the p-sylow subgroup of π is cyclic. Let π_p be a p-sylow subgroup, and let Φ_p be the group of automorphisms of π_p induced by inner automorphisms of π. Then the p-period of π is twice the order of Φ_p.*

The group Φ_p is, of course, isomorphic to $N(\pi_p)/C(\pi_p)$ where N and C denote the normalizer and centralizer, respectively.

Before proving these theorems, I will review some facts about the cohomology of groups. Suppose $h:\rho \to \pi$ is a monomorphism of finite groups. Then h induces a map of cohomology $h^*:\hat{H}^i(\pi, A) \to \hat{H}^i(\rho, A)$. Here A is a π-module and so can be regarded as a ρ-module by means of h. This map h^* is defined as follows. Let W be a Tate complex (or complete resolution in the terminology of [1, Ch. XII, §3]) for π. Then ρ acts on W through h, and W is ρ-free since h is a monomorphism. Thus W is also a Tate complex for ρ. The map h^* is now defined to be the map of cohomology induced by the inclusion $\mathrm{Hom}_\pi(W, A) \subset \mathrm{Hom}_\rho(W, A)$. In case h is an inclusion map, h^* is just the map $i(\rho, \pi)$ of [1, Ch. XII, §8]. Suppose $\pi \subset \Pi$, $x \in \Pi$ and $h:x\pi x^{-1} \to \pi$ is given by $h(y) = x^{-1}yx$. Then $h^*:\hat{H}^*(\pi, A) \to \hat{H}^*(x\pi x^{-1}, A)$ is just the map c_x of [1, Ch. XII, §8].

Let π' be a subgroup of π, and x an element of π. Then there are two obvious monomorphisms $i, f_x : \pi' \cap x\pi' x^{-1} \to \pi'$, namely, $i(y) = y$ and $f_x(y) =$

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