MARKOFF CHAINS AND MARTIN BOUNDARIES[1]

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In the latter half of [5] Doob extends to Markoff chains many results he had previously obtained for Brownian motions. Roughly, his argument rests on the theory of martingales and on properties of the Martin boundary established by R. S. Martin and M. Brelot using classical methods, the bridge between the two groundworks being the equivalence of resolutivity of the boundary and almost certain convergence of an appropriate Markoff chain.

This paper presents another argument, using only the basic properties of martingales and Markoff chains, in which the main convergence theorem of Doob is proved at the beginning by reversing the sense of time in a Markoff chain. I first intended to write a note giving the simple proof of this convergence by explicit calculation; the subject is so attractive, however, that I decided upon a brief complete exposition, including some material omitted from §17 of [7] about which I shall say a word.

In view of the symmetry of past and future in the notion of Markoff chain, the lack of such symmetry in defining Markoff chains with stationary transitions must puzzle many a probabilist. Now, a slight and momentarily ugly alteration of the latter definition yields the notion of random chain with approximately stationary transitions, a notion symmetric in past and future This symmetry is used in §2 to establish the convergence mentioned above and in §5 to reduce problems concerning the entrance boundary to ones concerning the exit boundary. The chains themselves are studied in §1 and in the first part of §5 in order to furnish the proper background for [5] and [7].

Doob's convergence theorem, established directly, leads to the proofs in §3 and §4 of the Poisson-Martin representation of excessive functions, the behavior of excessive functions near the Martin boundary, and the resolutivity of the Martin boundary. Of course, it is only the arrangement of material that distinguishes these sections.

Some remarks are deferred until §6, the last section, since most of them merely explain the departures from the language and definitions of Doob and Brelot.

Doob's argument and ours both hold for Brownian motions or, more generally, for the processes discussed in the third part of [7].

1. Random chains

The space of states is a countable set R which is provided with the discrete topology as a topological space and with the field of all its subsets as a meas-

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