A GENERAL ERGODIC THEOREM

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The purpose of this paper is to prove Theorem 1 which was conjectured by E. Hopf [4, p. 39].

The first operator theoretic generalization of the Birkhoff ergodic theorem was given by Doob [1] who proved that $\sum_{k=0}^{n} T^k f/n$ converges pointwise, where T is a Markov operator for which there is an invariant measure and f is the characteristic function of a set. It was noted by Kakutani [5] that Doob's method was applicable to give the same result for f merely a bounded function, and Doob [2] later applied the same method to obtain this convergence result for f in L_p, and more generally for $|f| \log^+ |f|$ in L_1 (relative to the invariant measure). Hopf [4] then proved the theorem assuming merely that f is integrable. Dunford & Schwartz [3] extended Hopf's result by dropping the assumption of positivity for T. More precisely, they dropped the restriction of assuming the operator to be a Markov operator and proved the theorem for an operator which does not increase the L_1 and L_∞ norms. Tsurumi [6] proved a result with the same conclusion as ours but with much stronger and somewhat complicated hypothesis. In our paper we drop the assumption that T does not increase the L_∞ norm, consider positive operators which do not increase the L_1 norm, and prove a theorem about ratios of sums of transforms of two functions. Our result implies the Hopf theorem. The Dunford and Schwartz theorem, although not implied by our theorem, can be obtained by modifying the method which we use in this paper.

Theorem 1. *Let T be a positive linear operator on L_1 of a positive measure space $(S, \mathcal{G}, \mu)$, and let T have L_1 norm less than or equal to one. Then if f and p are functions in L_1, and if p is nonnegative, the limit*

$$\lim_{n\to\infty} \sum_{k=0}^{n} T^k f \Big/ \sum_{k=0}^{n} T^k p$$

exists and is finite almost everywhere on the set $A = \{s : T^k p > 0 \text{ for some } k \geqq 0\}$.

Let $D_n(f, p) = \sum_{k=0}^{n} T^k f / \sum_{k=0}^{n} T^k p$ and suppose in what follows that T satisfies the conditions stated in the theorem. All given functions in what follows will be supposed to be in L_1, and all functions which are constructed will be in L_1 by construction (this will be obvious).

Lemma 1. *If $f = f^+ - f^-$, and if* $\sup \sum_{k=0}^{n} T^k f > 0$ *on a set B, then there exist sequences $\{d_k\}$ and $\{f_k\}$ of nonnegative functions such that*

(i) $\sum_{k=0}^{N} \int d_k + \int f_N \leqq \int f^+$,

(ii) $\sum_{k=0}^{\infty} d_k = f^-$ *on* B,

(iii) $T^N f^+ = \sum_{k=0}^{N} T^{N-k} d_k + f_N$.

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