ON THE NORM OF A GROUP

BY

EUGENE SCHENKMAN[1]

The object of this note is to simplify and improve on the results of Wos (cf. [3]) about the norm of a group. We prove the following theorem.

THEOREM. *The norm of a group is in the second center of the group; the centralizer of the norm includes the commutator subgroup of the group.*

Proof. It should be recalled that the norm $N(G)$ of a group G (cf. [1]) is the set of elements a such that for each subgroup H of G, $aH = Ha$. Thus the norm comprises precisely those elements a such that aga^{-1} is, for every g in G, a power of g. It follows that if $[a, g]$ designates the commutator $aga^{-1}g^{-1}$, then $[[a, g], g] = 1$ for all g in G. In particular, $[[a, ga], ga] = 1$. But

$$[[a, ga], ga] = [[a, g], ga] = [[a, g], g]g[[a, g], a]g^{-1}$$

(cf. [4], p. 80), and hence $[[a, g], a] = 1$, which implies that a commutes with its conjugates. Thus

(A) *Every element of the norm is in an Abelian normal subgroup of* G; *and* $[a, g^m] = [a, g]^m$ *for every integer* m.

On the basis of the above, we next show that

$$(1) \qquad [[h, a], g^{-1}] = [[g, a], h].$$

For $[a, g] = g^r$, a power of g; $[a, h] = h^s$, a power of h; and $[a, hg] = (hg)^t$, a power of hg. Furthermore $aga^{-1} = g^{r+1}$, $aha^{-1} = h^{s+1}$, $ahga^{-1} = (hg)^{t+1}$, whence it follows that $h^{s+1}g^{r+1} = (hg)^{t+1}$. But h^s and g^r commute because they are in the Abelian normal subgroup spanned by a and its conjugates. It follows that $hg^rh^sg = (hg)^{t+1}$ and

$$g^{r-1}h^sg = (hg)^t = [a, hg] = [a, h]h[a, g]h^{-1} = h^shg^rh^{-1}.$$

Then $h^{-s}g^{-1}h^sg = g^{-r}hg^rh^{-1}$, and since $[a, h]^{-1} = [h, a]$,

$$(1) \qquad [[h, a], g^{-1}] = [[g, a], h],$$

as was to be shown.

Now $[[g, a], h]$ is a power of h since a, and consequently $[g, a]$, belongs to the norm; and the above equation (1) shows that $[[g, a], h]$ is a power of g as well. Hence

(B) $[[g, a], h]$ *is in the center of the group* K *generated by* g *and* h.

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