THE RADIUS OF UNIVALENCE OF BESSEL FUNCTIONS I[1]

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1. Introduction

In this paper we begin a study of the radius of univalence of Bessel functions. It is necessary to normalize the function, and a natural form is

$$\tilde{J}_\nu(z) = z^{1-\nu} J_\nu(z) = a_1^{(\nu)} z - a_3^{(\nu)} z^3 + a_5^{(\nu)} z^5 - \cdots, \tag{1.1}$$

where the coefficients $a_{2m+1}^{(\nu)}$ are defined by the recurrence relation

$$a_1^{(\nu)} = 2^{-\nu}/\Gamma(1+\nu), \qquad a_{2m+1}^{(\nu)} = a_{2m-1}^{(\nu)}/4m(\nu+m), \qquad m = 1, 2, \cdots. \tag{1.2}$$

It is well known that $\tilde{J}_\nu(z)$ is an entire function for any ν. With respect to the normalization factor $z^{1-\nu}$, we note that it is unique in the sense that $1 - \nu$ is the only exponent for which $z^{1-\nu}J_\nu(z)$ is schlicht in some neighborhood of the origin when $\nu > -1$.

The index ν is assumed to be real.

We present here a complete solution for $\nu > -1$. In §5 we state some results for $\nu < -1$ which appear plausible in the light of our computational experiments; we expect to handle these in a later paper.

2. Some general properties of Bessel functions

We require various standard results from the theory of Bessel functions and one from the theory of conformal representation. These will be quoted with references, but without proof. We have quoted Watson [1], but the results will be found in many places, in particular in Erdélyi, Magnus, Oberhettinger, and Tricomi [2].

LEMMA 1. *For $\nu > -1$ the functions $J_\nu(z)$ and $\tilde{J}_\nu(z)$ have infinitely many zeros, and all are real.* Cf. Watson [1], pp. 478, 483.

As usual we shall denote the positive zeros of $\tilde{J}_\nu(z)$, in order of magnitude, as $j_{\nu,1} < j_{\nu,2} < j_{\nu,3} < \cdots$. We note that, in addition, 0 and $-j_{\nu,m}$ $(m = 1, 2, \cdots)$ are zeros of $\tilde{J}_\nu(z)$.

LEMMA 2. *For fixed m, $j_{\nu,m}$ is an increasing function of ν.* Cf. Watson [1], p. 508.

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