POTENTIALS AND THE RANDOM WALK

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1. Introduction

Given an integer $s \geqq 3$, write $e_1, e_2, \cdots, e_s$ for the s coordinate vectors $(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, 0, \cdots, 1)$, spanning the s-dimensional lattice of points with integral coordinates, and let s_n denote the position at time n $(= 0, 1, 2, \cdots)$ of a particle performing the standard s-dimensional random walk according to the following rule: fixing the first $n - 1$ steps $s_1, s_2, \cdots, s_{n-1}$, the particle starts afresh at s_{n-1}, jumping next to one of the $2s$ neighbors $s_n = s_{n-1} \pm e_1, s_{n-1} \pm e_2, \cdots, s_{n-1} \pm e_s$ of s_{n-1}, the chance of landing at a particular neighbor being $(2s)^{-1}$.

Given a set B of lattice points, the probability p_B that the random walk hits B at some time $n < +\infty$, as a function of the starting point of the walk, is *excessive* in the sense that $\mathbf{G}p_B \leqq 0$, where $\mathbf{G}$ is Laplace's difference operator:

1.1 $$(\mathbf{G}p)(a) = (2s)^{-1}\textstyle\sum_{k \leqq s, n=1,2} p(a + (-)^n e_k) - p(a).$$

B. H. Murdoch [1, pp. 13–19] proved that if $p \geqq 0$, and if $\mathbf{G}p = 0$, then p is constant,[2] and, with the help of this result, it follows, as Murdoch himself noted, that p_B is the sum of the *potential* $\mathbf{K}e_B$ and the constant $p_B(\infty)$, where $e_B = -\mathbf{G}p_B$ $(\geqq 0)$, $\mathbf{K}e_B$ is the expectation of $\sum_{n \geqq 0} e_B(s_n)$, as a function of the starting point of the walk, and $p_B(\infty)$ is the (constant) probability $P.(\mathbf{B})$ of the event $\mathbf{B}$ that $s_n \in B$ for an infinite number of integers n.

$P.(\mathbf{B})$ is either 0 or 1. When $P.(\mathbf{B}) = 0$, p_B is the greatest *potential* $p \leqq 1$ such that $\mathbf{G}p = 0$ outside B, and, on the strength of the example of the Newtonian potential in 3 dimensions, it is natural to think of e_B as the electrostatic distribution of charge on the conductor B and to introduce the total charge (of e_B) as the capacity $C(B)$ of B.

Given a set B, it is an interesting problem to decide whether $P.(\mathbf{B}) = 0$ or 1; the solution is

1.2 $$P.(\mathbf{B}) = 0 \text{ or } 1 \quad \text{according as} \quad \textstyle\sum_{n \geqq 0} 2^{-n(s-2)} C(B_n) < \text{ or } = +\infty,$$

where B_n is the intersection of B and the spherical shell $2^n \leqq |a| < 2^{n+1}$. Wiener's test for the singular points of the Newtonian electrostatic potential (see Courant and Hilbert [1, p. 286]) served us as a model, and for this reason we call 1.2 Wiener's test also. B. H. Murdoch [1, pp. 45–47] came close to proving 1.2 and used his method to compute $P.(\mathbf{B})$ for sets B similar to those

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[2] J. Capoulade [1] also stated this result and S. Verblunsky [1] and R. Duffin [1, pp. 242–245] proved it. Murdoch's results lie much deeper.