LIMITING THEOREMS FOR AGE-DEPENDENT BRANCHING PROCESSES[1]

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1. To motivate the theorems that will be stated and proved here, consider particles which are assumed to have a life span with cumulative probability distribution function $G(t)$. At the end of its life a particle is assumed to split into n particles with probability q_n, where each particle has the same properties as the original. It is assumed $q_n \geqq 0$ and $n \geqq 0$. The generating function associated with $\{q_n\}$ is

$$h(s) = \sum_{j=0}^{\infty} q_j s^j, \qquad h(1) = 1. \tag{1.1}$$

Given a particle at $t = 0$, let the probability that there are n particles at time $t \geqq 0$ be $p_n(t) \geqq 0$. The generating function is

$$F(s, t) = \sum_0^{\infty} p_j(t) s^j, \qquad F(1, t) = 1. \tag{1.2}$$

Then the above description suggests that $F(s, t)$ satisfies

$$F(s, t) = \int_{0-}^{t} h(F(s, t - y))\, dG(y) + s[1 - G(t)]\,. \tag{1.3}$$

This problem with $h(s) = s^2$ and with mild restrictions on $G(t)$ has been studied by Bellman and Harris [1]. References to the literature will be found in [1].

In the special case where $G(t)$ is a step function with one discontinuity, the process becomes the Galton-Watson branching process. For this case the author has shown [2] that a best possible condition on $h(s)$ for the desired limiting theorems to hold is just a little more stringent than the existence of the first moment

$$\mu = h'(1) = \sum_1^{\infty} j q_j < \infty. \tag{1.4}$$

It will be shown here that, with $\mu > 1$, essentially the same condition on $h(s)$ as given in [2] is sufficient to yield the basic limit theorem in the age-dependent case subject to restrictions on $G(t)$.

If, following [1], the random variable representing the number of particles at time t, starting with one particle at $t = 0$, is denoted by $Z(t)$, then for $t \geqq 0$ and $|s| \leqq 1$

$$\begin{aligned} F(s, t) &= E[s^{Z(t)}], \\ E[Z(t)] &= m(t) = \partial F(1, t)/\partial s. \end{aligned} \tag{1.5}$$

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