INVERSION OF TOEPLITZ MATRICES II

BY

HAROLD WIDOM

1. Introduction

With a function $\varphi(\theta) \in L_1(0, 2\pi)$, $\varphi(\theta) \sim \sum_{-\infty}^{\infty} c_k e^{ik\theta}$, is associated the semi-infinite Toeplitz matrix $T_\varphi = (c_{j-k})_{0 \leq j,k < \infty}$. In case $\sum |c_k| < \infty$, $T_\varphi$ represents a bounded operator on the space $l_1^\infty$ of bounded sequences

$$X = \{x_0, x_1, \ldots\},$$

and in [1] a necessary and sufficient condition was found for the invertibility of $T_\varphi$ (i.e., the existence of a bounded inverse for $T_\varphi$), namely that $\varphi(\theta) \neq 0$ and $\Delta_{-\pi \leq \theta \leq \pi} \arctan \varphi(\theta) = 0$. If $\varphi(\theta) \in L_\infty$, $T_\varphi$ represents a bounded operator on the space $l_2^\infty$ of square-summable sequences, and in §3 of [1] sufficient conditions were obtained for invertibility in this situation.

The purpose of the present paper is to obtain conditions which are necessary as well as sufficient for invertibility of $T_\varphi$ as an operator on $l_2^\infty$. That the situation is quite different in the $l_1^\infty$ and $l_2^\infty$ cases can be seen, for instance, from the fact that in the former, the set of $\varphi$ for which $T_\varphi$ is invertible forms a group, while in the latter we may have $T_\varphi$ invertible but $T_{\varphi^2}$ not (Corollary 2 of Theorem IV).

As in all problems of Wiener-Hopf type, and this is one, the basic idea is a certain type of factorization. In our case, the idea is that of writing $T_\varphi$ as the product of triangular Toeplitz matrices (which amounts to a factorization of $\varphi$), the question of invertibility for these being simpler since any two triangular Toeplitz matrices of the same type commute. Thus, roughly speaking, if $\varphi$ is sufficiently nice, we can factor $T_\varphi$ and then invert each factor, thus obtaining the inverse of $T_\varphi$. This gives rise to sufficient conditions for invertibility, as in [1, §3]. Now in the $l_1^\infty$ theory it turned out that the $\varphi$'s for which this could be carried out were exactly those giving rise to invertible Toeplitz matrices; thus the invertibility of $T_\varphi$ implies the existence of a suitable factorization of $\varphi$. It is the content of Theorem I of the present paper that this situation prevails also in the $l_2^\infty$ case. From this result we easily settle the invertibility question for triangular and self-adjoint Toeplitz matrices.

For general Toeplitz matrices we have been unable to find a simple criterion for invertibility; there is one however (Theorem IV) in case $\arctan \varphi(\theta)$ is reasonably well-behaved.

Before proceeding, we introduce some notation. For $f(\theta) \in L_\nu(0, 2\pi)$,