ON WEIGHTED DISTORTION IN CONFORMAL MAPPING

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1. A few years ago Komatu and Nishimiya [4] raised the question of obtaining bounds for the distortion in the spherical metric of normalized univalent functions in the unit circle, that is, for the quantity $|f'(z)|(1 + |f(z)|^2)^{-1}$ for a given value of $|z|$. The explicit values they obtained were for the most part not sharp. Quite recently Oikawa [6] obtained the best possible upper and lower bounds for all values of $|z|$, $0 < |z| < 1$ using the variational method. None of the preceding authors seems to have realized that the lower bound has essentially been known for many years. Indeed by the standard distortion theorem, if $|z| = r$, $0 \leqq r < 1$,

$$|f'(z)|^{-1} \leqq (1 + r)^3/(1 - r),$$

while by a result of Löwner [5] in a form given by Robinson [7]

$$|f(z)|^2 |f'(z)|^{-1} \leqq r^2/(1 - r^2).$$

By adding these,

$$(1 + |f(z)|^2)|f'(z)|^{-1} \leqq ((1 + r)^4 + r^2)/(1 - r^2),$$

which on inverting gives just the lower bound obtained by Oikawa. As is well known, equality occurs in this for the slit functions and for them only.

Of much more interest is the fact that the form of solution obtained by Oikawa has very little dependence on the explicit form of the spherical distortion. In fact an analogous result applies to expressions of the form $F(|f(z)|, |f'(z)|)$ whenever F satisfies certain fairly simple restrictions to be discussed in detail below.

2. As usual we denote by S the family of functions $f(z)$ regular and univalent for $|z| < 1$ with $f(0) = 0, f'(0) = 1$. We begin with a discussion of the functions which play the extremal role in our problems.

THEOREM 1. *Let* $0 < r < 1$, $r/(1 + r)^2 \leqq q \leqq r/(1 - r)^2$. *Then there exists a unique function* $f(z, r, q)$ *in* S *with* $f(r, r, q) = q$ *mapping* $|z| < 1$ *conformally onto an admissible domain* [2; 3, p. 49] *with respect to the quadratic differential for* $q \neq r$,

$$Q(w, a, q)dw^2 = \frac{q^2(w - a)dw^2}{aw^2(w - q)^2},$$

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