HOMOTOPY GROUPS, COMMUTATORS, AND Γ-GROUPS

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1. Introduction

In [7] J. H. C. Whitehead introduced for a simply connected complex K an exact sequence

$$\to H_{n+1}(K) \xrightarrow{\nu_{n+1}} \Gamma_n(K) \xrightarrow{\lambda_n} \pi_n(K) \xrightarrow{\mu_n} H_n(K) \to$$

(called Γ-sequence) involving the homotopy groups $\pi_n(K)$, the homology groups $H_n(K)$, and a new kind of groups $\Gamma_n(K)$, called the Γ-groups of K.

As was shown in [3], *homology groups are*, in a certain precise sense, *"obtained from the homotopy groups by abelianization."* The above exact sequence suggests that between Γ-groups and homotopy groups a dual relationship might exist. It is the purpose of this note to show that this is indeed the case, and that the Γ-groups are, in a similar sense, *"obtained from the homotopy groups by taking commutator subgroups."*

The result will be stated in terms of c.s.s. complexes and c.s.s. groups. We shall freely use the notation and results of [3] and [4].

The main step in the argument is a rather curious lemma on connected c.s.s. groups. It states that for a connected c.s.s. group F and any integer $n \geqq 2$, every n-simplex in the commutator subgroup of F is homotopic with an n-simplex in the commutator subgroup of the $(n-1)$-skeleton of F.

2. The main lemma

We shall state a lemma which describes a rather surprising property of connected c.s.s. groups. The lemma shows how connectedness, although its definition involves only 0-simplices and 1-simplices, influences quite strongly the behaviour of a c.s.s. group in all higher dimensions. This explains somewhat why connectedness is such a strong condition to impose on a c.s.s. group or, equivalently, (cf. [4], §§9 and 11) why simple connectedness is such a strong condition to impose on a CW-complex or a c.s.s. complex.

For another application of this lemma see [6].

Let F be a c.s.s. group; denote by $[F, F] \subset F$ the *commutator subgroup*, i.e., the (c.s.s.) subgroup such that $[F, F]_n = [F_n, F_n]$ for all n; and for every integer $s \geqq 0$ let $F^s \subset F$ be the *s-skeleton*, i.e., the smallest (c.s.s.) subgroup containing F_s. Then we have

LEMMA 2.1. *Let F be a connected c.s.s. group, and let $\sigma \in [F, F]_n$, where $n \geqq 2$. Then there exist elements*

$$\phi \in [F^{n-1}, F^{n-1}]_n \quad \textit{and} \quad \rho \in [F^n, F^n]_{n+1}$$

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