VARIOUS AVERAGING OPERATIONS ONTO SUBALGEBRAS[1]

BY
CHANDLER DAVIS[2]

The operations dealt with in this paper are functions defined on a self-adjoint algebra over the complex numbers and with values in a subalgebra of it. (The subalgebra is arbitrary.) They are called "averaging" because the formal properties which define them (Definition 2.1) are possessed by the conditional expectations of I. E. Segal.[3] If the subalgebra is just the constants, the averaging operations are exactly the states; if the subalgebra is the whole algebra, the only averaging operation is the identity. In case the whole algebra is commutative, the study of averagings has been carried very far by G. Birkhoff [2] and J. L. Kelley [10].

Another class of operations (for which I claim no novelty except the name) is introduced, and elementary properties set forth, in §3. I had begun studying this class for somewhat different reasons, but it turns out to be closely related to the averaging operations—indeed, to be a subclass, and that subclass which behaves least like the classical conditional expectations which occur in the abelian case. The main result of this paper is the expression (in §4) of an arbitrary averaging operation in terms of these two special types.

The effect of averaging operations on the spectrum of a hermitian operator is the subject of §§6–7. Theorem 7.2, a simple extension of a theorem of Hardy, Littlewood, and Pólya, may have independent interest.

All the algebras in this paper are finite-dimensional. Extension of some of the results to arbitrary von Neumann algebras is the idea which leads me often to express things in terms of algebras, commutors, etc., when some proofs would be a little shorter using only matrices. [*Added March* 31, 1959. The program of characterizing noncommutative conditional expectations, in analogy to the work of Birkhoff, Moy, and others in the commutative case, was initiated by M. Nakamura and his colleagues; see especially [13]. I regret that I was in ignorance of this work when I wrote the present paper. Their results deal with the infinite-dimensional case; they do not seem to contain my main results here as specializations.]

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[2] National Science Foundation Fellow.

[3] Further justification for the terminology appears later, e.g., Definition 3.4, Theorem 6.2, Theorem 4.1. No doubt "*positive* averaging operation" would be a better term; or "abstract expectation".