INVERSION OF TOEPLITZ MATRICES

BY

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1. Introduction

This paper deals with the inversion of the Toeplitz matrix $T = (c_{j-k})$, $j, k = 0, 1, \cdots$. It will be assumed that the c_k are the Fourier coefficients of a function $\varphi(\theta)$,

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\varphi(\theta)\, d\theta, \qquad k = 0, \pm 1, \cdots .$$

Since the inversion of T is equivalent to the solution of a system of equations of the form

$$\textstyle\sum_{k=0}^{\infty} c_{j-k}\, x_k = y_j, \qquad j = 0, \pm 1, \cdots ,$$

we see that we are dealing with the discrete analogue of a Wiener-Hopf equation. It might be expected then that we shall look for a factorization of φ of the form $\varphi = \varphi_+ \varphi_-$, where $\varphi_+(\theta)$ and $\varphi_-(\theta)$ are boundary values of functions analytic inside and outside the unit circle, respectively. This, in fact, is the crux of the matter.

In Section 2 we consider the case $\sum_{-\infty}^{\infty} | c_k | < \infty$. Then T may be considered a bounded operator on the space l_∞^+ of bounded sequences $X = \{x_0, x_1, \cdots\}$ with $\| X \| = \sup | x_k |$, and a necessary and sufficient condition is found for the invertibility of T (Theorem I). In case T is invertible, a generating function is found for the entries of the matrix T^{-1} (Theorem III). As a consequence of the theory we obtain a theorem of Tauberian type: Certain sets are shown to be fundamental in l_1^+, the space of all $X = \{x_0, x_1, \cdots\}$ with $\| X \| = \sum_0^\infty | x_k | < \infty$ (Corollary of Theorem II).

In Section 3 the condition $\sum | c_k | < \infty$ is dropped, but it is still assumed that φ is bounded. In this case T may be considered an operator (bounded by the boundedness of φ using Parseval's relation) on the space l_2^+ of square summable sequences $X = \{x_0, x_1, \cdots\}$ with $\| X \|^2 = \sum_{k=0}^{\infty} | x_k |^2$, and we find a sufficient condition for the invertibility of T (Theorem IV).

Note added in proof. A substantial part of this paper (Theorems I and II and an analogue of Theorem III) was discovered independently by M. G. KREĬN in his paper, *Integral equations on the half-line with a difference kernel*, Uspehi Mat. Nauk, vol 13, no. 5 (1958), pp. 3–120 (Russian). Where the operator T is concerned, with $\sum | c_k | < \infty$, our paper is practically identical with Kreĭn's, in regard to both methods and results. Kreĭn has gone further

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