MAXIMALLY UNCLEFT RINGS AND ALGEBRAS

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I. Definitions and elementary properties

1. We shall follow the definitions in [7] for semiprimary, primary, and completely primary rings. Our rings are semiprimary with nilpotent radical, and our algebras are of finite dimension over a base field F. On these rings and algebras we shall place an additional restriction which for commutative completely primary rings makes them exactly coefficient rings, not fields (see [4] for definition of coefficient ring).

In a ring A, with radical $N \subset A$ (proper inclusion), there may exist a subring A^* such that A^* is semisimple and maps onto A/N in the natural homomorphism of A onto A/N. A is called *cleft* if A^* exists, otherwise, *uncleft*. The existence of and relation between such subalgebras when A is an algebra and A/N is separable is known as the Wedderburn-Malcev Theorem [3]. When A is a complete equal characteristic local ring, the existence of A^* is due to I. S. Cohen [2]. (See also [4], [5], [9].) We shall say that the ring A is *maximally uncleft* (briefly, m.u.) if A/J is uncleft for every ideal J, $J \subset N$, $N \neq \{0\}$. The corresponding statement defining an m.u. algebra is of course relative to the base field.

Our main purpose here is to prove that (a) the study of m.u. rings and algebras reduces to a study of m.u. completely primary rings and algebras, (b) the presence of the m.u. property in algebras with central radicals is determined by the 2-dimensional cohomology groups of pure inseparable field extensions, and (c) the product of two division algebras is m.u. if and only if the product of their centers is m.u. Before proceeding, we note some examples of m.u. rings and algebras:

(1) $C/(p^r)$, where C is the ring of integers, p a prime, and $r > 1$.

(2) $F(\alpha)$, where α is algebraic over the field F with minimum function $(x^q - c)^n$, $q = p^e$, $n > 1$, $e > 0$, p the characteristic of F, $x^q - c$ irreducible over F.

(3) Any coefficient ring of an unequal characteristic complete local ring, hence of a commutative unequal characteristic completely primary ring.

(4) B_n, a primary matric ring with B an m.u. completely primary ring.

The second example is m.u. as an algebra over F, not as a ring.

2. We now give some convenient equivalent definitions and properties of an m.u. ring or algebra. We always assume $N \neq \{0\}$, $N \subset A$.

Let T denote a residue system (system of representatives) for the classes of A/N. Let (T) denote the subring of A generated by T. If A is an algebra,

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