ON FINITE GROUPS CONTAINING AN ELEMENT OF ORDER FOUR WHICH COMMUTES ONLY WITH ITS POWERS

BY

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This paper is a study of a particular case of the following question:

Let K be a subset of a finite group H. Suppose that another finite group G contains H in such a way that the centralizer in G of any element of K is contained in H. Then what can we say about the structure of G? In particular for given H and K, are there infinitely many *simple* groups G satisfying the above condition?

So far no general solution nor general method to attack this problem has been found. The purpose of this paper is to give an answer to this question in the very special case that H is a cyclic group of order 4 and K consists of its generators. The result of this investigation may be stated as follows:

Let G be a finite group containing an element π of order 4. If π commutes only with its powers, then either G contains a normal subgroup of index 2 which does not contain π, or G contains an abelian normal subgroup G_0 of odd order such that the factor group G/G_0 is one of the following groups: $SL(2, 3)$, $SL(2, 5)$, $LF(2, 7)$, or the alternating groups A_6 or A_7.

1. Throughout this paper G stands for a finite group which satisfies the following condition $(*)$:

$(*)$ G contains an element π of order 4 such that π commutes only with its own powers.

We shall use the letter V to denote the subgroup of G generated by π, and T stands for the subgroup of V generated by the involution $\tau = \pi^2$.

PROPOSITION 1. *Let S be a 2-Sylow subgroup of G containing V. Then S is generated by π and another element ρ with one of the following five relations*:

(1) $\rho^2 = 1, \quad \rho\pi\rho^{-1} = \pi^{-1}$;

(2) $\rho^2 = \tau, \quad \rho\pi\rho^{-1} = \pi^{-1}$;

(3) $\rho^{2m} = \tau \ (m \geqq 2), \quad \pi\rho\pi^{-1} = \rho^{-1}$;

(4) $\rho^{2m} = \tau \ (m \geqq 2), \quad \pi\rho\pi^{-1} = \rho^{-1}\tau$;

(5) $\rho = 1$.

Proof. By assumption the centralizer of V in S coincides with V. Hence V contains the center of S. In particular T is the only normal subgroup of order 2 in S. If $S = V$, we have the last case (5). We now assume that $S \neq V$. The normalizer of V is therefore of order 8. If $[S:e] = 8$, we have

Received February 10, 1958.

[1] This investigation has been done at Harvard University with support from the National Science Foundation.