A THEOREM ON AFFINE CONNEXIONS[1]

BY
N. HICKS

Introduction

The object of this study is to generalize a theorem of W. Ambrose (see [2]) to the non-Riemannian case.[2] The outline of our proof is the same as that of the theorem of Ambrose, but many features must be modified and are more complicated. We refer the reader to the bibliography for all definitions of fundamental concepts on C^∞ manifolds and connexions.

Our theorem characterizes a simply connected C^∞ manifold, on which is defined a complete affine connexion, by the behavior of the curvature and torsion forms under parallel translation along finitely broken geodesics emanating from some fixed point. In the analytic case, one need only consider unbroken geodesics. As an immediate consequence of our result, we obtain the (known) theorem which states that a simply connected manifold, on which is defined a complete connexion having zero curvature and torsion invariant under parallel translation, is a Lie group. Relaxing the simply connected hypothesis to just connected and adding an assumption that the holonomy group be the identity, we can prove the manifold is a homogeneous space, and an example shows we cannot hope to prove M is a Lie group under these hypotheses.

1. Notation and statement of the main theorem

Let M be a C^∞ manifold and m a point in M; then we denote the tangent space at m by M_m. Let $B(M)$ denote the bundle of bases over M, and π the map of $B(M)$ onto M. Given a connexion on $B(M)$, thus an "affine" connexion, we denote its 1-form by ω, i.e., ω is a C^∞ 1-form with values in $\mathrm{gl}(d, R)$, the Lie algebra of the general linear group $\mathrm{GL}(d, R)$, where d is the dimension of M and R is the field of real numbers. By a complete connexion we mean one in which our geodesics can be indefinitely extended (indefinitely in terms of the parameter, for the geodesic may be closed). A complete connexion on $B(M)$ allows us to define a map $\exp_m : M_m \to M$, for any $m \in M$, and a map $\mathrm{Exp}_b : M_m \to B(M)$, for any $b \in B(M)$. The latter allows us to carry information in $B(M)$ back to the simpler space M_m. We define these maps and denote the tangent field to a curve σ by T_σ.

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