DIFFERENTIAL OPERATORS WITH THE POSITIVE MAXIMUM PROPERTY

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1. Introduction

Let I be a fixed open interval, and consider a linear functional operator whose domain and range consist of real functions continuous in a subinterval of I. We say that f is in the local domain of Ω at the point $s \in I$, in symbols $f \in D(\Omega, s)$, if both f and Ωf are continuous in a neighborhood of s. Similarly, the global domain $D(\Omega, J)$ for an interval $J \subset I$ consists of the functions f such that both f and Ωf are continuous in J. We shall suppose that Ω is of *local character* in the following sense: if f vanishes identically in some neighborhood of s, then $f \in D(\Omega, s)$ and $\Omega f(s) = 0$. For such an operator the restriction of Ωf to an interval J depends only on the behavior of f in J.

We say that Ω has the *positive maximum property if one has* $\Omega f(s) \leqq 0$ *for each* $s \in I$ *and each* $f \in D(\Omega, s)$ *which attains a positive local maximum at* s. The classical differential operator defined by $af'' + bf' + cf$ with $a > 0$ has the positive maximum property if $c \leqq 0$. In view of the well-known role of such operators in many theories, we propose in this note to find a canonical form for the general operators of local character with the positive maximum property.

Let x be a continuous strictly increasing function in I, and m a strictly increasing right continuous function (not necessarily bounded). We shall view x as a scale, the increments of m as a measure on the Borel sets of I. The symbol $D_x f$ will be used indiscriminately for right and left derivatives provided they exist at each point and are continuous except for jumps. Differentiation with respect to m has the obvious meaning provided we agree to consider increments only for *closed* intervals. The operator $\Omega_0 = D_m D_x$ has been discussed in [1] as a natural generalization of the classical operator $aD_s^2 + bD_s$; it is characterized by the *strong* maximum property that $\Omega_0 f(x) \leqq 0$ for each x such that f attains a local maximum at x.

Operators of the form $\Omega_0 + c$ with $c \leqq 0$ have the positive maximum property, and we shall show that the most general operator with this property is of a similar, though slightly more intricate, form.

We parametrize I by x and consider an arbitrary *convex* continuous function ω of x; that is, $\omega > 0$ and $D_x \omega \uparrow$ throughout I. The operator A defined by

$$Af = \frac{1}{\omega} D_m \left\{ \omega^2 D_x \frac{f}{\omega} \right\} \tag{1.1}$$

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