THE ENGEL ELEMENTS OF A SOLUBLE GROUP

BY

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An ordered pair of subsets (A, B) of a given group G shall be said to satisfy the Engel condition if to each $a \in A$ and $b \in B$ there corresponds an integer $k = k(a, b)$ such that

$$[a, \underbrace{b, \cdots, b}_{k}] = 1.$$

(Here $[x_1, x_2]$ stands for the element $x_1^{-1}x_2^{-1}x_1 x_2$ and, for $k > 1$, $[x_1, \cdots, x_{k+1}] = [[x_1, \cdots, x_k], x_{k+1}]$.) The fact that (A, B) satisfies the Engel condition will be denoted by the symbol $A \mathfrak{e} B$. If, moreover, the integer k can be chosen independent of the element a in A, or of b in B, then this will be written $A \,|\mathfrak{e}\, B$, or $A \,\mathfrak{e}|\, B$, as the case may be.

We shall be mainly concerned with situations in which one of the sets A, B consists of a single element and the other of the whole group. It will be convenient to have a terminology specially adapted to these cases. Let us therefore call an element g a *left Engel element*,[1] or *bounded left Engel element* according as $G \mathfrak{e} g$, or $G \,|\mathfrak{e}\, g$; and a *right Engel element*, or *bounded right Engel element* according as $g \mathfrak{e} G$, or $g \,\mathfrak{e}|\, G$. Thus, for example, an element in a locally nilpotent normal subgroup is a left Engel element, and one in a nilpotent normal subgroup is a bounded left Engel element; while an element in any term of the ascending central series is a right Engel element, and one in a finite term of this series is a bounded right Engel element. Just how typical these examples really are is at present unknown. The best that can be said is the following: in a group, satisfying the maximal condition on subgroups, the set of all left Engel elements coincides with the set of all bounded left Engel elements and forms the maximal nilpotent normal subgroup; while the set of all right Engel elements coincides with the set of all bounded right Engel elements and forms the hypercentre of the group. These exceedingly elegant results are due to Reinhold Baer [3].

Our main aim in this paper is to show that, besides the groups with maximal condition, there exists another class of groups in which the Engel elements are well behaved: we shall, in fact, prove that in every soluble group the sets consisting of the four types of Engel elements all form subgroups. However, unlike the situation in the groups studied by Baer, it is here quite possible for no two of these subgroups to coincide: we shall construct a soluble group in which the four subgroups are distinct from each other and from the hypercentre. We begin, in §1, by considering four conditions analogous to, but considerably stronger than, the four Engel conditions, and we show that, in

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[1] Plotkin calls a left Engel element a nilelement.