CHARACTERIZATIONS OF THE ALGEBRA OF ALL REAL-VALUED CONTINUOUS FUNCTIONS ON A COMPLETELY REGULAR SPACE

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Introduction

If X is a topological space, then we denote by $C(X)$ the set of all real-valued continuous functions on X. For X compact,[1] the set $C(X)$ has been characterized from a variety of topologico-algebraic points of view.[2] For X an arbitrary completely regular space, however, no such characterization of $C(X)$ has previously been given. The object of this paper is to obtain several such characterizations of $C(X)$. (For a partial result in this direction see Shirota [10, Theorem 12].)

The first section is preliminary in nature. In §2 we represent certain rings A as subrings of $C(X)$, where X is a completely regular space uniquely determined by A. Similar results are obtained in §3 for A an algebra[3] and X a Q-space [5] and in §4 for A an algebra and X compact.

In order to characterize $C(X)$ for X an arbitrary completely regular space, it suffices [5] to assume that X is a Q-space. In §§5 and 6 we obtain such characterizations of $C(X)$, regarding $C(X)$ as an algebra, as a lattice-ordered algebra [2], and as a vector lattice [1]. Moreover, for X compact, we give, in §5, new characterizations of $C(X)$ as an algebra.

1. Some separation conditions

In this section we introduce and investigate briefly some separation properties of certain subsets of $C(X)$.

Let A be a subset of $C(X)$. We shall adopt the following definitions:

(1) A is *weakly pseudoregular* in case X has a subbase $\mathcal{U}$ of open sets such that for $U \in \mathcal{U}$ and $x \in U$ there is an $\alpha > 0$ in R and an $f \in A$ such that $|f(x) - f(y)| \geqq \alpha$ whenever $y \notin U$.

(2) A is *pseudoregular* (*regular*) in case (i) A contains the identity e of $C(X)$, and (ii) whenever $x \in X$ and U is an open neighborhood of x, there is an $f \in A$ such that $f(x) = 0$ and $f(y) \geqq 1$ ($f(y) = 1$) for all $y \notin U$.[4]

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[1] We shall assume that all compact [6] spaces are Hausdorff.

[2] For references to results of this type see for example [3] and [7].

[3] By an algebra we shall always mean an algebra A over the real field R. If A has an identity, we shall denote it by e. We shall also adopt the convention that lower case Greek letters denote elements of R, unless otherwise specified.

[4] The most natural definitions of "pseudoregular" and "regular" would omit requirement (i). Its presence, however, does not affect the generality of our results; we include it merely as a matter of terminological convenience.