RANDOM WALKS

BY

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Introduction

An important class of discrete Markoff chains are the random walks X_n, $n = 0, 1, 2, \cdots$, whose state space is a set of consecutive integers and whose one-step transition probabilities

$$P_{ij} = \Pr\{X_{n+1} = j \mid X_n = i\}$$

form a Jacobi matrix; that is, $P_{ij} = 0$ if $|i - j| > 1$. As transitions can occur only to the neighboring states, we may regard such a process as a discrete version of a continuous diffusion model.

We shall use the notation $P_{ii-1} = q_i$, $P_{ii} = r_i$, $P_{ii+1} = p_i$ throughout. The m-step transition probabilities

$$P_{ij}^{(m)} = \Pr\{X_{n+m} = j \mid X_n = i\}$$

form a matrix $P^{(m)}$ which satisfies

$$P^{(0)} = I, \qquad P^{(1)} = (P_{ij}) \equiv P,$$
$$P^{(m+1)} = P^{(m)}P = PP^{(m)},$$

so that $P^{(m)}$ is simply the product of m copies of P.

It is convenient to distinguish three cases according as the state space is the *finite* set $0, 1, \cdots, N$, or the *semi-finite* set $0, 1, \cdots, n, \cdots$, or the *doubly infinite* set $\cdots, -1, 0, 1, \cdots$. In the next two sections we discuss only the semi-infinite case, and the modifications necessary for the other two cases are presented in Section 3.

The "problem" of random walks may be described as follows: The fundamental matrix P is given, and it is required to relate qualitative properties of the Markoff process to qualitative properties of P and to compute various functionals of the process in terms of P.

In recent years numerous publications have appeared treating specialized aspects of random walk processes [5], [6], [7]. Our approach will be to obtain an integral representation for the transition matrix through which the probabilistic structure of the process may be analyzed. The integral representation involves a system of polynomials orthogonal with respect to a distribution $\psi(x)$ in the closed interval $[-1, 1]$.

A formalism which suggests the integral representation is as follows: Associ-

Received October 21, 1957.

[1] This work was supported by the Office of Naval Research.