REMARKS ON QUASI-FROBENIUS RINGS

BY
JEAN DIEUDONNÉ

1.

The theory of Frobenius or quasi-Frobenius rings has, from the start, been connected with the idea of "duality". But in most papers on that theory, by "duality" one understands, either the lattice-theoretic duality [1], [10], or, when the ring A under consideration is an algebra over a commutative ring K, the duality of *K-modules* [3], [8]. Actually, it seems to me that the kind of duality which is most closely related to these questions is the duality of *A-modules*; and I propose to show in this paper how very elementary considerations of duality theory can simplify and unify many known results on quasi-Frobenius rings, and give new characterizations for these rings.[1]

2. Modules with perfect duality

By a ring I will always understand an associative ring A having a unit element 1; all A-modules are supposed to be unitary. The elementary theory of duality [2, §4] associates to each left (right) A-module E its dual E^*, which is a right (left) A-module; further, to every submodule M of E (resp. E^*) is associated its *orthogonal* M^0, which is a submodule of E^* (resp. E); one has the trivial relations:

(i) $M \subset N$ implies $N^0 \subset M^0$, $\quad (M + N)^0 = M^0 \cap N^0$;

(ii) $M \subset M^{00}$, $\quad M^0 = M^{000}$.

In addition, the theory defines

(iii) a natural homomorphism $E \to E^{**}$;

(iv) a natural isomorphism $(E/M)^* \to M^0$;

(v) a natural monomorphism $E^*/M^0 \to M^*$,

M being an arbitrary submodule of E; moreover,

(vi) if E is a direct sum $M_1 + \cdots + M_n$, E^* is naturally identified to the direct sum $M_1^* + \cdots + M_n^*$ (M_i^* being identified by (iv) to the orthogonal of $\sum_{j \neq i} M_j$).

Finally, if A_s (resp. A_d) is A considered as left (resp. right) A-module,

(vii) $(A_s)^* = A_d$, $\quad (A_d)^* = A_s$,

Received November 14, 1957.

[1] *Added in proof.* After this paper was written, Professor A. Rosenberg kindly drew my attention to the following paper which I had overlooked:

K. MORITA AND H. TACHIKAWA, *Character modules, submodules of a free module, and quasi-Frobenius rings*, Math. Zeit., vol. 65 (1956), pp. 414–428.

In that paper, the authors study the quasi-Frobenius rings from the point of view of duality of A-modules, and prove a slightly weaker version of result (3.4) below (they assume that the duals of simple A-modules are simple), by essentially the same arguments as mine.