EXTENSIONS AND OBSTRUCTIONS FOR RINGS

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1. The setting of the problem

The cohomology theory of rings, in the form recently introduced [15], will here be shown appropriate to the systematic treatment of the general extension problem for rings.

The treatment is parallel to the known theory [5] of the extensions of groups. If G is a normal subgroup of a group E, the assignment to each $e \in E$ of the operation of conjugation by e in G induces a homomorphism θ of the quotient group $Q = E/G$ into the group of automorphism classes of G. The converse problem of group extensions therefore starts with the data: groups Q and G plus a homomorphism θ. These data are called a "Q-kernel" by Eilenberg-Mac Lane [5]. On the center C of G the homomorphism θ assigns to each element of Q a well defined automorphism of C; thus C may be regarded as a module over the integral group ring of Q, and the cohomology groups $H^n(Q; C)$ are then available. To θ one assigns an element of $H^3(Q, C)$ as "obstruction"; there exists an extension E of G by Q which realizes θ if and only if this obstruction is zero. When the obstruction is zero, the usual description of extensions by factor sets yields a one-one correspondence between $H^2(Q, C)$ and the set of those equivalence classes of extensions of G by Q which realize θ. These results [5] yield an algebraic interpretation of the two- and three-dimensional cohomology groups and provide a refinement of the usual extension theory (normally attributed to Schreier [17], but actually initiated[2] by Hölder [14]) in which the map θ and the factor sets are all treated together, in a somewhat indigestible lump.

There are subsequent and parallel studies for the extensions of associative algebras (Hochschild [11]) and of Lie algebras (Hochschild [12, 13]). In both cases, the algebras are taken over a field and hence have the additive structure of a vector space over that field. Consequently the extension problem for the additive structure involved is trivial, and only the multiplicative structure is substantially involved in the cohomology theory. The new cohomology for rings to be used here has as its object precisely the simultaneous treatment of additive and multiplicative structures. For example, Everett [10] has developed the analogue of the Schreier extension theory for the case of rings,

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