ON CASTELNUOVO'S CRITERION OF RATIONALITY $p_a = P_2 = 0$ OF AN ALGEBRAIC SURFACE

To Emil Artin on his sixtieth birthday

BY

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1. Introduction

Let F be a nonsingular (irreducible) algebraic surface over an algebraically closed ground field k. A theorem of Castelnuovo asserts that if the arithmetic genus p_a and the bigenus P_2 of F are both zero then F is a rational surface. This theorem has now been proved for fields k of arbitrary characteristic p, except in the case $(K^2) = 1$, where K is a canonical divisor on F.[2] In our cited paper MM (see footnote 2) we have stated that we have also a proof for the case $(K^2) = 1$, and in the present paper we shall give this proof.

An immediate consequence of Castelnuovo's criterion of rationality is the well-known theorem of Castelnuovo on the rationality of plane involution. This theorem, in the case of arbitrary characteristic, is to be stated as follows:

Let $k(x, y)$ be a purely transcendental extension of an algebraically closed field k, of transcendence degree 2, and let Σ be a field between k and $k(x, y)$, also of transcendence degree 2 over k.[3] *If $k(x, y)$ is a separable extension oj Σ, then Σ is a pure transcendental extension of k.*

We shall show by an example that the condition of separability of $k(x, y)/\Sigma$ is essential.

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We shall make use of results established in MM for the case of surfaces F for which $P_a = P_2 = 0$ and $(K^2) > 0$. If $(K^2) = 1$, then the Riemann-Roch inequality shows that the dimension of the anticanonical system $| K_a | (= | -K |)$ is $\geqq 1$. If $| K_a |$ is reducible, then F is rational, by Proposition 7.3 of MM. *We shall therefore assume that $| K_a |$ is irreducible.* In that case we have dim $| K_a | = 1$ (MM, Lemma 10.1), i.e., $| K_a |$ is a pencil; it has a single base point O, every member K_a of $| K_a |$ has a simple point at O, and

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[2] See our recent paper *The problem of minimal models in the theory of algebraic surfaces*, Amer. J. Math., vol. 80 (1958), pp. 146–184. This paper will be referred to in the sequel as MM.

[3] The theorem is also true if Σ/k has transcendence degree 1 (without any assumption on separability), but in that case the theorem is an easy consequence of the theorem of of Lüroth.