LOWEST ORDER EQUATION FOR ZEROS OF A HOMOGENEOUS LINEAR DIFFERENTIAL POLYNOMIAL

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Introduction

If $\mathfrak{F}$ is a field and x belongs to an algebraic extension of $\mathfrak{F}$, then the algebraic properties of x are completely determined by the irreducible polynomial over $\mathfrak{F}$ which vanishes at x. Similarly, if $\mathfrak{F}$ is an ordinary differential field (i.e., a field with given derivation) of characteristic zero and x belongs to a differentially algebraic differential field extension of $\mathfrak{F}$, the differential algebraic properties of x are completely determined by the irreducible differential polynomial $F(y) \in \mathfrak{F}\{y\}$ of lowest order which vanishes at x. We shall call $F(y)$, which is unique up to a nonzero factor in $\mathfrak{F}$, the *lowest differential polynomial* of x over $\mathfrak{F}$, and we shall call the differential equation $F(y) = 0$ the *lowest equation* for x over $\mathfrak{F}$.

Let $\mathfrak{F}$ be an ordinary differential field of characteristic zero, and let C, the field of constants of $\mathfrak{F}$, be algebraically closed. Let $(x_1, \cdots, x_n)$ be a fundamental system of zeros of a homogeneous linear differential polynomial $L_n(y) \in \mathfrak{F}\{y\}$ such that the field of constants of $\mathfrak{F}\langle x_1, \cdots, x_n\rangle$ is C. $\mathfrak{F}\langle x_1, \cdots, x_n\rangle$ is called a Picard-Vessiot extension of $\mathfrak{F}$ (hereafter denoted by P.V.E.), and the group G of automorphisms of $\mathfrak{F}\langle x_1, \cdots, x_n\rangle$ over $\mathfrak{F}$ can be identified with an algebraic group of linear transformations of the vector space V_n over C with basis $(x_1, \cdots, x_n)$. (See [3].) We sometimes call G the group of $L_n(y)$ over $\mathfrak{F}$.

It is the purpose of this paper to obtain information about G when the lowest equation over $\mathfrak{F}$ for some $x \in V_n$ is known, and about the lowest equation for every $x \in V_n$ when G is one of the classical groups.

Notation. Throughout this paper $\mathfrak{F}$ will stand for an ordinary differential field of characteristic zero whose field of constants C is algebraically closed. $L_n(y)$ will always stand for a homogeneous linear differential polynomial of order n. Whenever we speak of zeros of $L_n(y) \in \mathfrak{F}\{y\}$, we restrict ourselves to zeros which belong to a P.V.E. of $\mathfrak{F}$. We shall therefore be able to say, for some $L_n(y) \in \mathfrak{F}\{y\}$, that every one of its zeros satisfies a differential equation over $\mathfrak{F}$ of lower order. If, for a given $L_n(y) \in \mathfrak{F}\{y\}$, there exist an integer r and $L_r(y)$, $L_{n-r}(y) \in \mathfrak{F}\{y\}$ such that $1 \leqq r \leqq n-1$ and

$$L_n(y) = L_{n-r}(L_r(y)),$$

we say that $L_n(y)$ is *composite* over $\mathfrak{F}$, that $L_n(y)$ is *the composite* of $L_r(y)$ and $L_{n-r}(y)$, and that $L_n(y)$ is *decomposable* by $L_r(y)$ on the *right*. If an

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