THE PROBABILITY THAT A MATRIX BE NILPOTENT

BY
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In this paper we determine the number of nilpotent n by n matrices over (i) a finite field of characteristic p, and (ii) the integers modulo m. The results are most simple when expressed as probabilities by dividing by the total number of matrices in each case.

THEOREM 1. *The probability that an n by n matrix over* $GF(p^\alpha)$ *be nilpotent is* $p^{-\alpha n}$.

Proof. Let A be an n by n nilpotent matrix over the finite field F. Then[2] $V_n(F)$ has a basis $\{v_s^i\}$, $i = 1, \cdots, k$; $s = 1, \cdots, r_i$, such that

$$v_s^i A = v_{s-1}^i \qquad (1 \leqq i \leqq k;\ 1 \leqq s \leqq r_i), \tag{1}$$

where it is understood that $v_0^i = 0$. Associated with each such A there is a partition π of n,

$$\pi{:}n = r_1 + r_2 + \cdots + r_k \qquad (r_1 \geqq r_2 \geqq \cdots \geqq r_k \geqq 1),$$

and two matrices are similar if and only if their corresponding partitions are identical. Let $g(\pi)$ be the number of matrices in the similarity class determined by π. Then the probability of nilpotence is

$$P = p^{-\alpha n^2} \sum\nolimits_\pi g(\pi).$$

To determine $g(\pi)$, we select and fix a representative A of the similarity class belonging to π, together with a basis $\{v_s^i\}$ associated with A by (1). We then transform A by the ν nonsingular matrices over F to obtain all the elements of the class, each with multiplicity μ, where μ is the number of nonsingular matrices which commute with A. Then $g(\pi) = \nu/\mu$. Now it is known[3] that

$$\nu = x^{-n^2} f(n),$$

where $x = p^{-\alpha}$ and

$$f(n, x) = f(n) = (1 - x)(1 - x^2) \cdots (1 - x^n) \qquad (n \geqq 1)$$
$$f(0) = 1.$$

It remains to determine μ.

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[2] See, for example, A. A. ALBERT, *Modern higher algebra*, University of Chicago Press, 1937, Chapter 4.
[3] L. E. DICKSON, *Linear groups*, Leipzig, 1901, p. 77.